

Gaussian optimizers and the additivity problem in quantum information theory

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Abstract

We give a survey of the two remarkable analytical problems of quantum information theory. The main part is a detailed report of the recent (partial) solution of the quantum Gaussian optimizers problem which establishes an optimal property of Glauber's coherent states – a particular instance of pure quantum Gaussian states. We elaborate on the notion of quantum Gaussian channel as a noncommutative generalization of Gaussian kernel to show that the coherent states, and under certain conditions only they, minimize a broad class of the concave functionals of the output of a Gaussian channel. Thus, the output states corresponding to the Gaussian input are “the least chaotic”, majorizing all the other outputs. The solution, however, is essentially restricted to the gauge-invariant case where a distinguished complex structure plays a special role.

We also comment on the related famous additivity conjecture, which was solved in principle in the negative some five years ago. This refers to the additivity or multiplicativity (with respect to tensor products of channels) of information quantities related to the classical capacity of quantum channel, such as $(1 \rightarrow p)$ -norms or the minimal von Neumann or Rényi output entropies. A remarkable corollary of the present solution of the quantum Gaussian optimizers problem is that these additivity properties, while not valid in general, do hold in the important and interesting class of the gauge-covariant Gaussian channels.

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1 Introduction

The quantum Gaussian optimizers problem is an analytical problem that arose in quantum information theory at the end of past century, and which has an independent mathematical interest. Only recently a solution was found [23], [53] in a considerably common situation, while in full generality the problem still remains open. To explain the nature and the difficulty of the problem we start from the related classical problem of Gaussian maximizers which has been studied rather exhaustively, see Lieb [50] and references therein. Consider an integral operator G from $L_p(\mathbb{R}^s)$ to $L_q(\mathbb{R}^r)$ given by a Gaussian kernels (i.e. exponential of a quadratic form) with the $(q \rightarrow p)$ – norm

$$\|G\|_{q \rightarrow p} = \sup_{f \neq 0} \|Gf\|_p / \|f\|_q = \sup_{\|f\|_q \leq 1} \|Gf\|_p. \quad (1)$$

Under certain broad enough assumptions concerning the quadratic form defining the kernel, and also p and q , this operator is correctly defined, and the supremum in (1) is attained on Gaussian f . Moreover, under some additional restrictions any maximizer is Gaussian. As it is put in the title of the paper [50]: “Gaussian kernels have only Gaussian maximizers”.

Knowledge that the maximizer is Gaussian can be used to compute exact value of the norm (1); in fact a starting point of the classical Gaussian maximizers works were the result of K.I. Babenko [5] and a subsequent paper of Beckner [6] which established the best constant in the Hausdorff-Young inequality concerning the $(p \rightarrow p')$ -norm, $(p^{-1} + (p')^{-1})^{-1} = 1$, $1 < p \leq 2$, of the Fourier transform (which is apparently given by a degenerate imaginary Gaussian kernel).

A difficulty in the optimization problem (1) is that it requires *maximization* of a convex function, so the general theory of convex optimization is not of great use here (it only implies that a maximizer of $\|Gf\|_p$ belongs to a face of the convex set $\|f\|_q \leq 1$). Instead, the solution is based on substantial use of the classical Minkowski’s inequality and the related multiplicativity of the classical $(q \rightarrow p)$ -norms with respect to tensor products of the integral operators.

A notable application of these classical results to a problem in quantum mathematical physics was Lieb’s solution [51] of Wehrl’s conjecture [63]. Let ρ be a density operator in a separable Hilbert space \mathcal{H} representing state of a quantum system; the “classical entropy” of the state ρ is defined as ¹

$$H_{cl}(\rho) = - \int_{\mathbb{C}} p_{\rho}(z) \log p_{\rho}(z) \frac{d^2 z}{\pi},$$

where $p_{\rho}(z) = \langle z | \rho | z \rangle$ is the diagonal value of the kernel of ρ in the system of Glauber’s coherent vectors² $\{|z\rangle; z \in \mathbb{C}\}$ [44], [33]. The conjecture was that $H_{cl}(\rho)$ has the minimal value if ρ is itself a coherent state i.e. projector onto one of the coherent vectors. Lieb [51] used exact constants in the Hausdorff-Young inequality for L_p -norms of Fourier transform [5], [6] and the Young inequality for convolution [6] to prove similar maximizer conjecture for $f(x) = x^p$ and considered the limit $\lim_{p \downarrow 1} (1 - p)^{-1} (1 - x^p) = -x \log x$.

¹ Throughout the paper the base of logarithm is a fixed number $a > 1$. In information theory the natural choice is $a = 2$, then all the entropic quantities are measured in “bits”.

² In analysis, they correspond to complex-parametrized Gaussian wavelets. Notice that this is the only place in the present article where we formally used Dirac’s notations, uncommon among mathematicians.

Recently, Lieb and Solovej [52], by using a completely different approach based on study of the spin coherent states, strengthened the result of [51] by showing that the coherent states minimize *any* functional of the form $\int_{\mathbb{C}} f(p_{\rho}(z)) \frac{d^2 z}{\pi}$, where $f(x), x \in [0, 1]$ is a nonnegative concave function with $f(0) = 0$.

In the language of quantum information theory, the affine map $G : \rho \rightarrow p_{\rho}(z)$, taking density operators ρ (quantum states) into probability densities $p_{\rho}(z)$ (classical states), is a “quantum-classical channel” [39]. Moreover, it transforms Gaussian density operators ρ (in the sense defined below in Sec. 3.1) into Gaussian probability densities, and in this sense it is a “Gaussian channel”. From this point of view, Wehrl entropy $H_{cl}(\rho)$ is the output entropy of the channel, and Lieb’s result says that it is minimized by pure Gaussian states ρ . Moreover, the corresponding result for $f(x) = x^p$ can be interpreted as “Gaussian maximizer” statement for the norm $\|G\|_{1 \rightarrow p}$. Notice that the case $q = 1$, which is excluded in the classical problem for obvious reasons, appears and is the most relevant in the quantum (noncommutative) case.

The quantum Gaussian optimizers problem described in the present paper refers to Bosonic Gaussian channels – a noncommutative analog of Gaussian Markov kernels and, similarly, requires maximization of convex functions (or minimization of concave functions, such as entropy) of the output state of the channel, while the argument is the input state. A general conjecture is that the optimizers belongs to the class of pure Gaussian states. The conjecture, first formulated in [42] in the context of quantum information theory, however natural it looks, resisted numerous attacks for several years. Among others, notable achievements were the exact solution for the classical capacity of pure loss channel [21] and a proof of additivity of the Rényi entropies of integer orders p [24] for special channels models. Even restricted to the class of Gaussian input states, the optimization problem turns out to be nontrivial [56], [31]. There was some hope that in solving the problem, similarly to Wehrl’s conjecture, one could also use the classical “Gaussian maximizers” results. However the solution found recently by Giovannetti, Holevo, Garcia-Patron [23], and Mari, Giovannetti, Holevo [53] uses completely different ideas based on a thorough study of structural properties of quantum Gaussian channels. As it was mentioned, a solution of the classical problem uses the Minkowski inequality and the implied multiplicativity of $(q \rightarrow p)$ -norms. However, the noncommutative analog of the Minkowski’s inequality [12] is not powerful enough to guarantee the multiplicativity of

norms (or additivity of the corresponding entropic quantities). Moreover, the related long-standing additivity problem in quantum information theory [34] was recently shown to have negative solution in general [26]. We show that, remarkably, a solution of the quantum Gaussian optimizers problem given in [23] implies also a proof of the multiplicativity/additivity property in the restricted class of gauge-covariant or contravariant quantum Gaussian channels.

It would then be interesting to investigate a possible development of such an approach to obtain noncommutative generalizations of the classical “Gaussian maximizers” results for $(q \rightarrow p)$ -norms. Such generalization could shed a new light to the hypercontractivity problem for quantum dynamical semigroups and related noncommutative analogs of logarithmic Sobolev inequalities, see e.g. [62].

2 The additivity problem for quantum channels

2.1 Definition of channel

Let \mathcal{H} be a separable complex Hilbert space, $\mathfrak{L}(\mathcal{H})$ the algebra of all bounded operators in \mathcal{H} and $\mathfrak{T}(\mathcal{H})$ the ideal of trace-class operators. The space $\mathfrak{T}(\mathcal{H})$ equipped with the trace norm $\|\cdot\|_1$ is Banach space, which is useful to consider as a noncommutative analog of the space L_1 . The convex subset of $\mathfrak{T}(\mathcal{H})$

$$\mathfrak{S}(\mathcal{H}) = \{\rho : \rho^* = \rho \geq 0, \text{Tr} \rho = 1\},$$

is a base of the positive cone in $\mathfrak{T}(\mathcal{H})$. Operators ρ from $\mathfrak{S}(\mathcal{H})$ are called *density operators* or *quantum states*. The state space is a convex set with the extreme boundary

$$\mathfrak{P}(\mathcal{H}) = \{\rho : \rho \geq 0, \text{Tr} \rho = 1, \rho^2 = \rho\}.$$

Thus extreme points of $\mathfrak{S}(\mathcal{H})$, which are called *pure states*, are one-dimensional projectors, $\rho = P_\psi$ for a vector $\psi \in \mathcal{H}$ with unit norm, see, e.g. [55].

The class of maps we will be interested is a noncommutative analog of Markov maps (linear, positive, normalized maps) in classical analysis and probability. Let $\mathcal{H}_A, \mathcal{H}_B$ be the two Hilbert spaces, which will be called input and output space, correspondingly. A map $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is *positive*

if $X \geq 0$ implies $\Phi[X] \geq 0$, and it is completely positive [61], [54] if the maps $\Phi \otimes \text{Id}_{(d)}$ are positive for all $d = 1, 2, \dots$, where $\text{Id}_{(d)}$ is the identity map of the algebra $\mathfrak{L}_d = \mathfrak{L}(\mathbb{C}^d)$ of complex $d \times d$ -matrices. Equivalently, for every nonnegative definite block matrix $[X_{jk}]_{j,k=1,\dots,d}$ the matrix $[\Phi[X_{jk}]]_{j,k=1,\dots,d}$ is nonnegative definite.

A linear map Φ is trace-preserving if $\text{Tr}\Phi[X] = \text{Tr}X$ for all $X \in \mathfrak{T}(\mathcal{H}_A)$.

Definition *Quantum channel* is a linear completely positive trace-preserving map $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$. Letter A will be always associated with the *input* of the channel, while B with the *output*. Sometimes, to abbreviate notations, we will write simply $\Phi : A \rightarrow B$. ■

Apparently, every channel is a positive map taking states into states: $\Phi[\mathfrak{S}(\mathcal{H}_A)] \subseteq \mathfrak{S}(\mathcal{H}_B)$. Since $\mathfrak{T}(\mathcal{H})$ is a base-normed space, this implies [17] that Φ is a bounded map from the Banach space $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_B)$. The dual Φ^* of the map Φ is uniquely defined by the relation

$$\text{Tr}\Phi[X]Y = \text{Tr}X\Phi^*[Y]; \quad X \in \mathfrak{T}(\mathcal{H}_A), Y \in \mathfrak{L}(\mathcal{H}_B), \quad (2)$$

and it is called *dual channel*. The dual channel is linear completely positive $*$ -weakly continuous map from $\mathfrak{L}(\mathcal{H}_B)$ to $\mathfrak{L}(\mathcal{H}_A)$, which is *unital*: $\Phi[I_{\mathcal{H}_B}] = I_{\mathcal{H}_A}$. Here and in what follows I with possible index denotes the unit operator in the corresponding Hilbert space.

There are positive maps that are not completely positive, a basic example provided by matrix transposition $X \rightarrow X^\top$ in a fixed basis.

From the definition of complete positivity one easily derives [39] that composition of channels $\Phi_2 \circ \Phi_1$ defined as

$$\Phi_2 \circ \Phi_1[X] = \Phi_2[\Phi_1[X]],$$

and naturally defined tensor product of channels

$$\Phi_1 \otimes \Phi_2 = (\Phi_1 \otimes \text{Id}_2) \circ (\text{Id}_1 \otimes \Phi_2)$$

are again channels.

2.2 Stinespring-type representation

The notion of completely positive map was introduced by Stinespring [61] in a much wider context of C^* -algebras. This allows also to cover the notion of

hybrid channel where the input is quantum while the output is classical or vice versa. An example of such channel was mentioned in Sec. 1. We will not pursue this topic further here, see [39], but only mention that complete positivity reduces to positivity in such cases.

Motivated by the famous Naimark's dilation theorem, Stinespring established a representation for completely positive maps of C*-algebras which in the case of quantum channel reduces [39] to

Proposition 1 *Let $\Phi : A \rightarrow B$ be a channel. There exist a Hilbert space \mathcal{H}_E and an isometric operator $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$, such that*

$$\Phi[\rho] = \text{Tr}_E V \rho V^*; \quad \rho \in \mathfrak{T}(\mathcal{H}_A), \quad (3)$$

where Tr_E denotes partial trace with respect to \mathcal{H}_E . The representation (3) is not unique, however any two representations with $V_1 : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_{E_1}$ and $V_2 : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_{E_2}$ are related via partial isometry $W : \mathcal{H}_{E_1} \rightarrow \mathcal{H}_{E_2}$ such that $V_2 = (I_B \otimes W) V_1$ and $V_1 = (I_B \otimes W^*) V_2$.

Consider a representation (3) for the channel Φ ; the *complementary channel* [37], [48] is then defined by the relation

$$\tilde{\Phi}[\rho] = \text{Tr}_B V \rho V^*; \quad \rho \in \mathfrak{T}(\mathcal{H}_A). \quad (4)$$

From the relation between the different representations (3), it follows that the complementary channel is unique in the following sense: any two channels $\tilde{\Phi}_1, \tilde{\Phi}_2$ complementary to Φ are isometrically equivalent in the sense that there is a partial isometry $W : \mathcal{H}_{E_1} \rightarrow \mathcal{H}_{E_2}$ such that

$$\tilde{\Phi}_2[\rho] = W \tilde{\Phi}_1[\rho] W^*, \quad \tilde{\Phi}_1[\rho] = W^* \tilde{\Phi}_2[\rho] W, \quad (5)$$

for all ρ . It follows that the initial projector $W^* W$ satisfies $\tilde{\Phi}_1[\rho] = W^* W \tilde{\Phi}_1[\rho]$, i.e. its support contains the support of $\tilde{\Phi}_1[\rho]$, while the final projector $W W^*$ has similar property with respect to $\tilde{\Phi}_2[\rho]$. The complementary to complementary can be shown isometrically equivalent to the initial channel, so that $\Phi, \tilde{\Phi}$ are called mutually complementary channels.

In general, we will say that two density operators ρ and σ (possibly acting in different Hilbert spaces) are *isometrically equivalent* if there is a partial isometry W such that $\rho = W \sigma W^*$, $\sigma = W^* \rho W$. Apparently, this is the case if and only if nonzero spectra (counting multiplicity) of the density operators ρ and σ coincide. We denote this fact with the notation $\rho \sim \sigma$. We have just shown that $\tilde{\Phi}_1[\rho] \sim \tilde{\Phi}_2[\rho]$ for arbitrary ρ .

Lemma 2 *Let $\tilde{\Phi}$ be a complementary channel (4), then $\Phi[P_\psi] \sim \tilde{\Phi}[P_\psi]$ for all $\psi \in \mathcal{H}_A$.*

Proof. Let $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ be the isometry from the representations (3), (4), then $\rho_{BE} = VP_\psi V^*$ is a pure state in $\mathcal{H}_B \otimes \mathcal{H}_E$, and the statement follows from a basic result in quantum information theory (“Schmidt decomposition”): if ρ_{BE} is a pure state in $\mathcal{H}_B \otimes \mathcal{H}_E$ and $\rho_B = \text{Tr}_E \rho_{BE}$, $\rho_E = \text{Tr}_B \rho_{BE}$ are its partial states, then $\rho_B \sim \rho_E$ (see e.g. Proposition 3 in [34]) ■

A different name for channel is dynamical map – in nonequilibrium quantum statistical mechanics they arise as irreversible evolutions of an open quantum system interacting with an environment [39]. Assume that there is a composite quantum system $AD = BE$ in the Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_D \simeq \mathcal{H}_B \otimes \mathcal{H}_E, \quad (6)$$

which is initially prepared in the state $\rho_A \otimes \rho_D$ and then evolves according to the unitary operator U . Then the output state ρ_B depending on the input state $\rho_A = \rho$ is

$$\Phi_B[\rho] = \text{Tr}_E U(\rho \otimes \rho_D)U^*, \quad (7)$$

while the output state of the “environment” E is the output of the channel

$$\Phi_E[\rho] = \text{Tr}_B U(\rho \otimes \rho_D)U^*. \quad (8)$$

If the initial state of D is pure, $\rho_D = P_{\psi_D}$, then by introducing the isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_D$, which acts as

$$V\psi = U(\psi \otimes \psi_D), \quad \psi \in \mathcal{H}_A,$$

we see that the relations (7), (8) convert into (3), (4), and Φ_E is just the complementary of Φ_B . Notice also that both partial trace and unitary evolution are completely positive operators, hence the maps (7), (8) are completely positive; vice versa, any quantum channel has a representation of such a form, see, e.g. [39].

Vast literature is devoted to study of quantum dynamical semigroups (noncommutative analog of Markov semigroups) and quantum Markov processes. Stinespring-type representation (3) underlies dilations of quantum dynamical semigroups to the unitary dynamics of open quantum system interacting with an environment [17], [35].

2.3 Entropic quantities and additivity

Consider the norm of the map Φ defined similarly to (1):

$$\|\Phi\|_{1 \rightarrow p} = \sup_{X \neq 0} \|\Phi[X]\|_p / \|X\|_1 = \sup_{\|X\|_1 \leq 1} \|\Phi[X]\|_p, \quad (9)$$

where $\|\cdot\|_p$ is the Schatten p -norm [55]. As shown in [4],

$$\|\Phi\|_{1 \rightarrow p}^p = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A)} \text{Tr} \Phi[\rho]^p = \sup_{\psi \in \mathcal{H}_A} \text{Tr} \Phi[P_\psi]^p, \quad (10)$$

where the second equality follows from convexity of the function $x^p, p > 1$.

The quantum Rényi entropy of order $p > 1$ of a density operator ρ is defined as

$$R_p(\rho) = \frac{1}{1-p} \log \text{Tr} \rho^p = \frac{p}{1-p} \log \|\rho\|_p, \quad (11)$$

Define the *minimal output Rényi entropy* of the channel Φ

$$\check{R}_p(\Phi) = \inf_{\rho \in \mathfrak{S}(\mathcal{H})} R_p(\Phi[\rho]) = \frac{p}{1-p} \log \|\Phi\|_{1 \rightarrow p} \quad (12)$$

and the *minimal output von Neumann entropy*

$$\check{H}(\Phi) = \inf_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi[\rho]). \quad (13)$$

In the limit $p \rightarrow 1$ the quantum Rényi entropies monotonely nondecreasing converge to the von Neumann entropy

$$\lim_{p \rightarrow 1} R_p(\rho) = -\text{Tr} \rho \log \rho = H(\rho).$$

In finite dimensions the set of quantum states is compact, hence by Dini's Lemma the minimal output Rényi entropies converge to the minimal output von Neumann entropy³.

Multiplicativity of the norm (9) for some channels Φ_1, Φ_2 ,

$$\|\Phi_1 \otimes \Phi_2\|_{1 \rightarrow p} = \|\Phi_1\|_{1 \rightarrow p} \cdot \|\Phi_2\|_{1 \rightarrow p} \quad (14)$$

is equivalent to the additivity of the minimal output Rényi entropies

$$\check{R}_p(\Phi_1 \otimes \Phi_2) = \check{R}_p(\Phi_1) + \check{R}_p(\Phi_2). \quad (15)$$

³The corresponding statement is not valid for infinite-dimensional channels (even for classical channels with countable set of states), M. E. Shirokov, private communication.

Closely related is the similar property for the minimal output von Neumann entropy:

$$\check{H}(\Phi_1 \otimes \Phi_2) = \check{H}(\Phi_1) + \check{H}(\Phi_2). \quad (16)$$

In finite dimensions, the validity of (15) for certain channels Φ_1, Φ_2 and p close to 1 implies (16) for these channels.

In the last two relations the inequality \leq (similarly to the inequality \geq in (14)) is obvious because the right-hand side is equal to the infimum over the subset of product states $\rho = \rho_1 \otimes \rho_2$. On the other hand, existence of “entangled” pure states which are not reducible to product states, is the cause for possible violation of the equality for quantum channels.

2.4 The channel capacity

The practical importance of the additivity property (16) is revealed in connection with the notion of the channel capacity. To explain it we assume that $\mathcal{H}_A, \mathcal{H}_B$ are finite dimensional for the moment.

For a quantum channel Φ , a noncommutative analog of the Shannon capacity, which we call χ -*capacity*, is defined by

$$C_\chi(\Phi) = \sup_{\{\pi_j, \rho_j\}} \left(H \left(\Phi \left[\sum_j \pi_j \rho_j \right] \right) - \sum_j \pi_j H(\Phi[\rho_j]) \right), \quad (17)$$

where the supremum is over all *quantum ensembles*, that is finite collections of states $\{\rho_1, \dots, \rho_n\}$ with corresponding probabilities $\{\pi_1, \dots, \pi_n\}$. The quantity (17) is closely related to the capacity $C(\Phi)$ of quantum channel Φ for transmitting classical information [34]. The *classical capacity* of a quantum channel is defined as the maximal transmission rate per use of the channel, with coding and decoding chosen for increasing number n of independent uses of the channel

$$\Phi^{\otimes n} = \underbrace{\Phi \otimes \dots \otimes \Phi}_n$$

such that the error probability goes to zero as $n \rightarrow \infty$ (for a precise definition see [39]). A basic result of quantum information theory, HSW Theorem [32], says that such defined capacity $C(\Phi)$ is related to $C_\chi(\Phi)$ by the formula

$$C(\Phi) = \lim_{n \rightarrow \infty} (1/n) C_\chi(\Phi^{\otimes n}).$$

Since $C_\chi(\Phi)$ is easily seen to be superadditive (i. e., $C_\chi(\Phi_1 \otimes \Phi_2) \geq C_\chi(\Phi_1) + C_\chi(\Phi_2)$), one has $C(\Phi) \geq C_\chi(\Phi)$. However if the additivity

$$C_\chi(\Phi_1 \otimes \Phi_2) = C_\chi(\Phi_1) + C_\chi(\Phi_2) \quad (18)$$

holds for a given channel $\Phi_1 = \Phi$ and an arbitrary channel Φ_2 , then

$$C_\chi(\Phi^{\otimes n}) = nC_\chi(\Phi), \quad (19)$$

implying

$$C(\Phi) = C_\chi(\Phi). \quad (20)$$

The reason for possible violation of the equality here, as well as in the cases (14), (15), (16), is existence of entangled states, which are not reducible to product states, at the input of tensor product channel $\Phi^{\otimes n}$.

2.5 Main conclusions

Thus it was natural to ask: does the the additivity property (16) holds globally, i.e. for tensor product of *any* pair of quantum channels Φ_1, Φ_2 ? The problem can be traced back to [8], see also [34]. Quite remarkably, Shor [60], see also [19], had shown the equivalence of the global properties of additivity of the χ - capacity and of the minimal output entropy.

Theorem 3 [60] *The properties (18) and (16) are globally equivalent in the sense that if one of them holds for all channels Φ_1, Φ_2 , then another is also true for all channels.*

The additivity is proved rather simply for all classical channels (see e.g. [15]), but in the quantum case the question remained open for a dozen of years, and was ultimately solved in the negative.

The detailed history of the problem up to 2006 can be found in [34], and here we only sketch the basic steps and the final resolution. In [1] it was suggested to approach the additivity property (16) via multiplicativity (14) of the $(1 \rightarrow p)$ -norms (equivalent to additivity (15) of the minimal output Rényi entropies). The first explicit example where this property breaks for $d = \dim \mathcal{H} \geq 3$ and large enough p was *transpose-depolarizing channel* [64]:

$$\Phi(\rho) = \frac{1}{d-1} [I \operatorname{Tr} \rho - \rho^\top], \quad (21)$$

where $\rho \in \mathfrak{L}_d$ is a matrix and ρ^\top its transpose. In particular, (15) with $\Phi_1 = \Phi_2 = \Phi$ fails to hold for $p \geq 4,7823$ if $d = 3$ (nevertheless, the additivity of $\tilde{H}(\Phi)$ and of $C_\chi(\Phi)$ holds for this channel). Five years later came important findings of Winter [65] and Hayden [28], see also [29], who showed existence of a pair of channels breaking the additivity of the minimal output Rényi entropy for all values of the parameter $p > 1$. The method of these and subsequent works is random choice of the channels, which for fixed dimensions are parametrized by isometries V in the representation (3), as well of the input states of the channels, combined with sufficiently precise probabilistic estimates for the norms (10). For finite dimensions the corresponding parametric sets are compact, and one usually takes the uniform distribution. Basing on this progress, Hastings [26] gave a proof of existence of channels breaking the additivity conjecture (16) corresponding to $p = 1$, in very high dimensions. Moreover, the probability of violation of the additivity tends to 1 as the dimensionalities tend to infinity. Hastings gave only a sketch, and the detailed proof following his approach was given by Fukuda, King and Moser [18], and further simplified by Brandao and M. Horodecki [10]. Later Szarek et al. [3] proposed a proof related to the *Dvoretzky-Mil'man theorem* on almost Euclidean sections of high-dimensional convex bodies.

Although, combined with theorem 3 this gives a definite negative answer to the additivity conjectures, several important issues remain open. All the proofs use the technique of random unitary channels or random states and as such are not constructive: they prove only existence of counterexamples but do not allow to actually produce them. Attempts to give estimates for the dimensions in which nonadditivity can happen based on Hastings' approach has led to overwhelmingly high values: the detailed estimates made in [18] gave $d \approx 10^{32}$ breaking the additivity by a quantity of the order 10^{-5} . The best result in this direction obtained in [7] states that “violations of the additivity of the minimal output entropy, using random unitary channels and a maximally entangled state, can occur if and only if the output space has dimension at least 183. Almost surely, the defect of additivity is less than $\log 2$, and it can be made as close as desired to $\log 2$ ”.

While this does not exclude possibility of better estimates, based perhaps on a different (but yet unknown) models, it casts doubt onto finding concrete counterexamples by computer simulation of random channels. From this point of view, the following explicit example given in [25] is of interest.

Consider the completely positive map

$$\rho \longrightarrow \Phi_-[\rho] = \text{Tr}_2 P_- \rho P_-, \quad \rho \in \mathfrak{T}(\mathcal{H} \otimes \mathcal{H}),$$

where P_- is the projector onto the antisymmetric subspace \mathcal{H}_- of $\mathcal{H} \otimes \mathcal{H}$ which has the dimensionality $\frac{d(d-1)}{2}$, and the partial trace is taken with respect to the second copy of \mathcal{H} . Its restriction to the operators with support in the subspace \mathcal{H}_- is trace preserving, hence it is a channel. It can be shown [39] that $\Phi_- = \frac{(d-1)}{2} \tilde{\Phi}^*$ where $\tilde{\Phi}^*$ is the dual to the complementary of the channel (21). For this simple channel the minimal Rényi entropies are nonadditive for all $p > 2$ and sufficiently large d , but unfortunately it is not clear if it could be extended to the most interesting range $p \geq 1$.

Coming back to arbitrary channels, it remains unclear what happens in small dimensions: perhaps the additivity still holds generically for some unknown reason, or its violation is so tiny that it cannot be revealed by numerical simulations. This is indeed surprising in view of the fact that the physical reason for nonadditivity is entanglement between the inputs of the parallel quantum channels, see [39] for more detail.

On the other hand, these results stress the importance of continuing efforts to find special cases where the additivity holds for some reason, and can be established analytically.

A survey of the main classes of such “additive” channels acting in finite dimensions was presented in [34]; below we briefly list the most important classes of channels Φ for which the additivity properties (16), (18) and (15) for $p > 1$ were established with $\Phi = \Phi_1$ and arbitrary Φ_2 .

- Qubit unital channels, i.e channels $\Phi : \mathfrak{L}_2 \rightarrow \mathfrak{L}_2$ satisfying $\Phi[I] = I$ [46]. Strikingly, there is still no analytical proof of the additivity for nonunital qubit channels, in spite of a convincing numerical evidence [27].
- Depolarizing channel in \mathfrak{L}_d :

$$\Phi[\rho] = (1 - p) \rho + p \frac{I}{d} \text{Tr} \rho, \quad 0 \leq p \leq \frac{d^2}{d^2 - 1},$$

which is the only unitarily-covariant channel, and can be regarded as noncommutative analog of completely symmetric channel in classical information theory [15]. The additivity properties (16), (15), (18) were proved by King [47].

- Entanglement-breaking channels. In finite dimensions these are channels of the form

$$\Phi[\rho] = \sum_j \rho_B \operatorname{Tr} \rho M_A,$$

where $\{M_A\}$ is a resolution of the identity in \mathcal{H}_A : $M_A \geq 0$, $\sum_j M_A = I_A$, and $\rho_B \in \mathfrak{S}(\mathcal{H}_B)$ (see [43]).

For the finite-dimensional entanglement-breaking channels the additivity of the minimal output von Neumann entropy and of the χ -capacity was established by Shor [59] and the additivity of the minimal output Rényi entropies – by King [45]. The additivity properties of entanglement-breaking channels were generalized to infinite dimensions by Shirokov [58].

- Complementary channels.

The additivity of the minimal output entropy is equivalent for a channel Φ and its complementary $\tilde{\Phi}$, see Lemma 4 below. The class of channels complementary to entanglement-breaking contains the Schur-multiplication maps of matrices $\rho = [c_{jk}]_{j,k=1,\dots,d}$ in \mathfrak{L}_d :

$$\tilde{\Phi}[\rho] = [\gamma_{jk} c_{jk}]_{j,k=1,\dots,d},$$

where $[\gamma_{jk}]_{j,k=1,\dots,d}$ is a nonnegative definite matrix such that $\gamma_{jj} \equiv 1$. For these channels, which are also called “Hadamard channels” the additivity of the χ -capacity was also established [48].

In the next Sections we consider Bosonic Gaussian channels which act in infinite-dimensional spaces. One of the main goals of the present paper is to show that the additivity holds for a wide class of gauge co- or contravariant Gaussian channels, i.e. those which respect a fixed complex structure in the underlying symplectic space.

2.6 Majorization for quantum states

From now on we again allow the Hilbert spaces in question to be infinite-dimensional. Denote by \mathfrak{F} the class of real concave functions f on $[0, 1]$, such that $f(0) = 0$. For any $f \in \mathfrak{F}$ and for any density operator ρ we can consider the quantity

$$\operatorname{Tr} f(\rho) = \sum_j f(\lambda_j),$$

where λ_j are the (nonzero) eigenvalues of the density operator ρ , counting multiplicity. Note that this quantity is defined unambiguously with values in $(-\infty, \infty]$. This follows from the fact that $f(x) \geq cx$, where $c = f(1)$, hence $\text{Tr}f(\rho) \geq c\text{Tr}\rho = c$. We also will use the fact that the functional $\rho \rightarrow \text{Tr}f(\rho)$ is (strictly) concave on $\mathfrak{S}(\mathcal{H})$ if f is (strictly) concave (see e.g. [11]).

Denote by $\lambda_j^\downarrow(\rho)$ the eigenvalues of a density operator ρ , counting multiplicity, arranged in the nonincreasing order. One says that density operator ρ *majorizes* density operator σ if

$$\sum_{j=1}^k \lambda_j^\downarrow(\rho) \geq \sum_{j=1}^k \lambda_j^\downarrow(\sigma), \quad k = 1, 2, \dots$$

A consequence of a well known result, see e.g. [11], is that this is the case if and only if $\text{Tr}f(\rho) \leq \text{Tr}f(\sigma)$ for all $f \in \mathfrak{F}$.

For a quantum channel Φ we introduce the quantity

$$\check{f}(\Phi) = \inf_{\rho \in \mathfrak{S}(\mathcal{H})} \text{Tr}f(\Phi[\rho]) = \inf_{P_\psi \in \mathfrak{P}(\mathcal{H})} \text{Tr}f(\Phi[P_\psi]), \quad (22)$$

where the second equality follows from the concavity of the functional $\rho \rightarrow \text{Tr}f(\Phi[\rho])$ on $\mathfrak{S}(\mathcal{H})$. Moreover, for strictly concave f , any minimizer is of the form P_ψ for some vector $\psi \in \mathcal{H}$.

In particular, taking $f(x) = -x \log x$ and $f(x) = -x^p$, we obtain $\check{f}(\Phi) = \check{H}(\Phi)$ and $\check{f}(\Phi) = -\|\Phi\|_{1 \rightarrow p}^p$.

Lemma 4 *For complementary channels, $\check{f}(\Phi) = \check{f}(\tilde{\Phi})$. Hence $\|\Phi\|_{1 \rightarrow p} = \|\tilde{\Phi}\|_{1 \rightarrow p}$, $\check{H}(\Phi) = \check{H}(\tilde{\Phi})$, $\check{R}_p(\Phi) = \check{R}_p(\tilde{\Phi})$, and the multiplicativity (14), as well as the additivity of the minimal output entropies (16), (15) holds simultaneously for pairs of channels Φ_1, Φ_2 and $\tilde{\Phi}_1, \tilde{\Phi}_2$.*

Proof. From Lemma 2, $\Phi[P_\psi]$ and $\tilde{\Phi}[P_\psi]$ have identical nonzero spectrum ($\Phi[P_\psi] \sim \tilde{\Phi}[P_\psi]$). Then

$$\text{Tr}f(\Phi[P_\psi]) = \text{Tr}f(\tilde{\Phi}[P_\psi]) \quad (23)$$

since $f(0) = 0$. Using second equality in (22) implies $\check{f}(\Phi) = \check{f}(\tilde{\Phi})$.

The statement about multiplicativity (additivity) then follows from the fact that the channel $\tilde{\Phi}_1 \otimes \tilde{\Phi}_2$ is complementary to $\Phi_1 \otimes \Phi_2$. ■

3 Quantum Gaussian systems

3.1 Gaussian states and channels

A real vector space Z equipped with a nondegenerate skew-symmetric form $\Delta(z, z')$ is called *symplectic space*. In what follows Z is finite-dimensional, in which case its dimensionality is necessarily even, $\dim Z = 2s$ [49]. A basis $\{e_j, h_j; j = 1, \dots, s\}$ in which the form $\Delta(z, z')$ has the matrix

$$\Delta = \text{diag} \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]_{j=1, \dots, s} \quad (24)$$

is called *symplectic*. The *Weyl system* in a Hilbert space \mathcal{H} is a strongly continuous family $\{W(z); z \in Z\}$ of unitary operators satisfying the *Weyl-Segal canonical commutation relation (CCR)*

$$W(z)W(z') = \exp\left[-\frac{i}{2}\Delta(z, z')\right]W(z + z'). \quad (25)$$

Thus $z \rightarrow W(z)$ is a projective representation of the additive group of Z . We always assume that the representation is irreducible. The Stone-von Neumann uniqueness theorem says that such a representation is unique up to unitary equivalence. It is well-known, see e.g. [55], that there is a family of selfadjoint operators $z \rightarrow R(z)$ with a common essential domain \mathcal{D} such

that

$$W(z) = \exp i R(z),$$

moreover, for any symplectic basis $\{e_j, h_j; j = 1, \dots, s\}$

$$R(z) = \sum_{j=1}^s (x_j q_j + y_j p_j)$$

on \mathcal{D} , where $R(e_j) = q_j$, $R(h_j) = p_j$, and $[x_1, y_1, \dots, x_s, y_s]$ are coordinates of vector z in the basis. Here the *canonical observables* $q_j, p_j; j = 1, \dots, s$ are selfadjoint operators in \mathcal{H} satisfying the Heisenberg CCR on \mathcal{D}

$$[q_j, p_k] \subseteq i\delta_{jk}I, \quad [q_j, q_k] = 0, \quad [p_j, p_k] = 0. \quad (26)$$

In physics the symplectic space is the phase space of the classical system (such as electro-magnetic radiation modes in the cavity), the quantum version of

which is described by CCR. Then s is number of degrees of freedom, or “normal modes” of the classical system.

The state given by density operator ρ in \mathcal{H} is called *Gaussian*, if its *quantum characteristic function*

$$\phi(z) = \text{Tr} \rho W(z)$$

has the form

$$\phi(z) = \exp \left(i m(z) - \frac{1}{2} \alpha(z, z) \right), \quad (27)$$

where m is a real linear form and α is a real bilinear symmetric form on Z . A necessary and sufficient condition for (27) to define a state is nonnegative definiteness of the (complex) Hermitian form⁴ $\alpha(z, z') - \frac{i}{2} \Delta(z, z')$ on Z or, briefly:

$$\alpha \geq \frac{i}{2} \Delta. \quad (28)$$

We will agree that the matrix of a bilinear form in fixed a symplectic base is denoted by the same letter, then (28) can be understood as inequality for Hermitian matrices, where α is real symmetric and Δ is real skew-symmetric.

A Gaussian state is pure if and only if α is a minimal solution of this inequality, see e.g. [38]. Operator J in Z is called *operator of complex structure* if

$$J^2 = -I, \quad (29)$$

where I is the identity operator in Z , and the bilinear form $\Delta(z, Jz')$ is an (Euclidean) inner product in Z , i.e.

$$\Delta(z, Jz') = \Delta(z', Jz) (= -\Delta(Jz, z')); \quad (30)$$

$$\Delta(z, Jz) \geq 0, \quad z \in Z. \quad (31)$$

The following characterization can be found in [16], [39]:

Proposition 5 *The minimal solutions of the inequality (28) are in one-to-one correspondence with the operators J of complex structure in Z given by the relation*

$$\alpha(z, z') = \frac{1}{2} \Delta(z, Jz'); \quad z, z' \in Z.$$

⁴ A complex-valued real-bilinear form $\beta(z, z')$ on Z will be called Hermitian if $\beta(z', z) = \overline{\beta(z, z')}$.

In this way to every complex structure corresponds the family of pure Gaussian states (27) with different values of m which are called the *J-coherent states*. The state with $m = 0$ is called *J-vacuum*. Let ρ_0 be a vacuum, then any associated coherent state is of the form $W(z')\rho_0W(z')^*$, as follows from the relation

$$W(z')^*W(z)W(z') = \exp[-i\Delta(z, z')]W(z)$$

and from nondegeneracy of the form $\Delta(z, z')$ due to which $m(z) = \Delta(z, z'_m)$.

Operator S in Z is called *symplectic* if $\Delta(Sz, Sz') = \Delta(z, z')$ for all $z, z' \in Z$. The unitary operators $W(Sz)$ satisfy the CCR (25) hence by the Stone-von Neumann uniqueness theorem there is a unitary operator U_S in \mathcal{H} such that

$$W(Sz) = U_S^*W(z)U_S, \quad z \in Z.$$

The map $S \rightarrow U_S$ is a projective representation of the group of all symplectic transformations in Z , sometimes called “metaplectic representation” [2] as it can be extended to a faithful unitary representation of the metaplectic group which is two-fold covering of the symplectic group.

Similarly, T is *antisymplectic* if $\Delta(Tz, Tz') = -\Delta(z, z')$ for all $z, z' \in Z$. There is an antiunitary operator U_T in \mathcal{H} such that

$$W(Tz) = U_T^*W(z)U_T, \quad z \in Z.$$

Let Z_A, Z_B be two symplectic spaces with the corresponding Weyl systems. Consider a channel $\Phi : A \rightarrow B$. The channel is called *Gaussian* if the dual channel satisfies

$$\Phi^*[W_B(z)] = W_A(Kz) \exp \left[il(z) - \frac{1}{2}\mu(z, z) \right], \quad z \in Z_B, \quad (32)$$

where $K : Z_B \rightarrow Z_A$ is a linear operator, l a linear form and μ is a real symmetric form on Z_B . In terms of characteristic functions of states,

$$\phi_B(z) = \phi_A(Kz) \exp \left[il(z) - \frac{1}{2}\mu(z, z) \right].$$

It follows that Gaussian channel maps Gaussian states into Gaussian states. A converse statement also holds true [16].

A necessary and sufficient condition on parameters (K, l, μ) for complete positivity of the map Φ is (see [14]) nonnegative definiteness of the Hermitian form

$$z, z' \rightarrow \mu(z, z') - \frac{i}{2} [\Delta_B(z, z') - \Delta_A(Kz, Kz')]$$

on Z_B , or, in matrix terms (if some bases are chosen in Z_A, Z_B),

$$\mu \geq \frac{i}{2} [\Delta_B - K^t \Delta_A K], \quad (33)$$

where t denotes transposition of a matrix. The proof using explicit construction of the representation of type (7) is given in [14], see also [39]; below in Proposition 11 below we give such a construction for an important particular class of Gaussian channels.

We call the Gaussian channel extreme⁵ if μ is a minimal solution of the inequality (33). This terminology stems from the fact that the minimality of μ is necessary and sufficient for the channel Φ to be an extreme point in the convex set of all channels with fixed input and output spaces [38].

Additivity hypothesis for quantum Gaussian channels: The additivity properties (15), (16) hold for any pair of Gaussian channels Φ_1, Φ_2 .

Hypothesis of quantum Gaussian minimizers: For any function $f \in \mathfrak{F}$ the infimum in (22) is attained on a pure Gaussian state ρ .

Any Gaussian channel has the covariance property

$$\Phi[W_A(z)\rho W_A(z)^*] = W_B(K^s z)\Phi[\rho]W_B(K^s z)^* \quad (34)$$

where K^s is the symplectic adjoint operator defined by the relation

$$\Delta_B(K^s z_A, z_B) = \Delta_A(z_A, K z_B).$$

It follows that the value $\text{Tr} f(\Phi[\rho])$ is the same for all coherent states $W(z)\rho_0 W(z)^*$ associated with a vacuum state ρ_0 .

These two problems turn out to be closely related. In what follows we describe positive solution for both of them in a particular and important class of Gaussian channels with gauge symmetry. However both conjectures remain open for general quantum Gaussian channels.

3.2 Complex structures and gauge symmetry

Given an operator of the complex structure J one defines in Z the Euclidean inner product $j(z, z') = \Delta(z, Jz')$. Then one can define in Z the structure

⁵ In quantum optics one speaks of *quantum-limited* channels [20].

of s -dimensional unitary space \mathbf{Z} in which iz corresponds to Jz and the (Hermitian) inner product⁶ is

$$\mathbf{j}(\mathbf{z}, \mathbf{z}') = \frac{1}{2}[\Delta(z, Jz') + i\Delta(z, z')] = \frac{1}{2}[j(z, z') - ij(z, Jz')].$$

From (29), (30) it follows that J is symplectic, that is $\Delta(Jz, Jz') = \Delta(z, z')$ for all $z, z' \in Z$. With every complex structure one can associate the cyclic one-parameter group of symplectic transformations $\{e^{\varphi J}; \varphi \in [0, 2\pi)\}$ which we call the *gauge group*. Hence, by the Stone-von Neumann uniqueness theorem, the gauge group in Z induces the one-parameter unitary group of the *gauge transformations* $\{U_\varphi; \varphi \in [0, 2\pi)\}$ in \mathcal{H} according to the formula

$$W(e^{\varphi J}z) = U_\varphi^* W(z) U_\varphi. \quad (35)$$

For the future use it will be convenient to introduce the complex parametrization of the Weyl operators by defining the *displacement operators*

$$D(\mathbf{z}) = W(Jz), \quad \mathbf{z} \in \mathbf{Z}. \quad (36)$$

A state ρ is gauge invariant if $\rho = U_\varphi \rho U_\varphi^*$ for all φ , which is equivalent to the property $\phi(z) = \phi(e^{\varphi J}z)$ of the characteristic function. In particular, Gaussian state (27) is gauge invariant if $m(z) \equiv 0$ and $\alpha(z, z') = \alpha(Jz, Jz')$. By introducing the Hermitian inner product in \mathbf{Z}

$$\boldsymbol{\alpha}(\mathbf{z}, \mathbf{z}') = \frac{1}{2}[\alpha(z, z') - i\alpha(z, Jz')],$$

we have $\boldsymbol{\alpha}(\mathbf{z}, \mathbf{z}) = \frac{1}{2}\alpha(z, z)$ since $\alpha(z, Jz')$ is skew-symmetric; moreover, the condition (28) is equivalent to nonnegative definiteness of the Hermitian form $\boldsymbol{\alpha}(\mathbf{z}, \mathbf{z}') - \frac{1}{2}\mathbf{j}(\mathbf{z}, \mathbf{z}')$ on \mathbf{Z} :

$$\boldsymbol{\alpha} \geq \frac{1}{2}\mathbf{j}. \quad (37)$$

This follows from application of the following Lemma to the form

$$z, z' \longrightarrow \beta(z, z') = \alpha(z, z') - \frac{i}{2}\Delta(z, z').$$

The relation (37) can be considered as the inequality for the matrices of the form, provided a basis is chosen in \mathbf{Z} . In an orthonormal basis, $\mathbf{j} = \mathbf{I}$ is the unit matrix.

⁶In accordance with convention accepted in mathematical physics, the inner product is complex linear with respect to \mathbf{z}' and anti-linear with respect to \mathbf{z} .

Lemma 6 *Let $\beta(z, z')$ be a bilinear complex-valued Hermitian form on real vector space Z , satisfying $\beta(Jz, Jz') = \beta(z, z')$, where J is a linear operator such that $J^2 = -I$. Then $\beta(z, z')$ is nonnegative definite i.e.*

$$\sum_{jk} \bar{c}_j c_k \beta(z_j, z_k) \geq 0 \quad (38)$$

for any finite collection $\{z_j\} \subset Z$ and any $\{c_j\} \subset \mathbb{C}$, if and only if

$$\operatorname{Re}\beta(z, z) \pm \operatorname{Im}\beta(z, Jz) \geq 0 \quad \text{for all } z \in Z. \quad (39)$$

Proof. (39) \implies (38): We have $\beta(z, z') = \operatorname{Re}\beta(z, z') + i\operatorname{Im}\beta(z, z')$, where $\operatorname{Im}\beta(z, z')$ is skew-symmetric, hence $\operatorname{Im}\beta(z, z) = 0$. By using the fact that $\beta(Jz, z') = -\beta(z, Jz')$ we obtain that also $\operatorname{Re}\beta(z, Jz')$ is skew-symmetric, hence $\operatorname{Re}\beta(z, Jz) = 0$. Thus

$$\operatorname{Re}\beta(z, z) \pm \operatorname{Im}\beta(z, Jz) = \beta(z, z) \mp i\beta(z, Jz).$$

Now introduce complexification $z \leftrightarrow \mathbf{z}$ by letting $Jz \leftrightarrow i\mathbf{z}$ and define two Hermitian forms on the complexification \mathbf{Z} of Z :

$$\beta^\mp(\mathbf{z}, \mathbf{z}') = \beta(z, z') \mp i\beta(z, Jz'). \quad (40)$$

Then β^- is sesquilinear i.e. complex linear with respect to \mathbf{z}' and anti-linear with respect to \mathbf{z} , while β^+ is anti-sesquilinear. From (40), (39),

$$\beta^\mp(\mathbf{z}, \mathbf{z}) = \operatorname{Re}\beta(z, z) \pm \operatorname{Im}\beta(z, Jz) \geq 0 \quad \text{for all } \mathbf{z} \in \mathbf{Z}, \quad (41)$$

hence by (anti-)sesquilinearity

$$\sum_{jk} \bar{c}_j c_k \beta^\mp(\mathbf{z}_j, \mathbf{z}_k) \geq 0.$$

By adding the two inequalities corresponding to plus and minus, we get (38).

Conversely, (38) \implies (39): Applying (38) to the collection $\{z_j, Jz_j\} \subset Z, \{c_j, \pm ic_j\} \subset \mathbb{C}$ we obtain

$$\sum_{jk} \bar{c}_j c_k [\beta(z_j, z_k) \pm i\beta(z_j, Jz_k)] \geq 0,$$

hence the forms (40) are nonnegative definite. By (anti-)sesquilinearity of these forms, this is equivalent to (41) i.e. (39). ■

Assume that in Z_A, Z_B operators of complex structure J_A, J_B are fixed, and let U_ϕ^A, U_ϕ^B be the corresponding gauge operators in $\mathcal{H}_A, \mathcal{H}_B$ acting according (35). Channel $\Phi : A \rightarrow B$ is called *gauge-covariant*, if

$$\Phi[U_\phi^A \rho (U_\phi^A)^*] = U_\phi^B \Phi[\rho] (U_\phi^B)^* \quad (42)$$

for all input states ρ and all $\phi \in [0, 2\pi]$. For the Gaussian channel (32) with parameters (K, l, μ) this reduces to

$$l(z) \equiv 0, \quad KJ_B - J_A K = 0, \quad \mu(z, z') = \mu(J_B z, J_B z').$$

The relation (32) for gauge-covariant Gaussian channel takes the form

$$\Phi^*[D_B(\mathbf{z})] = D_A(\mathbf{K}\mathbf{z}) \exp[-\boldsymbol{\mu}(\mathbf{z}, \mathbf{z})], \quad \mathbf{z} \in \mathbf{Z}_B, \quad (43)$$

where

$$\boldsymbol{\mu} \geq \pm \frac{1}{2} [\mathbf{j}_B - \mathbf{K}^* \mathbf{j}_A \mathbf{K}] \quad (44)$$

The equivalence of (44) and (33) is obtained by applying the lemma 6 to the Hermitian form

$$\beta(z, z') = \mu(z, z') - \frac{i}{2} [\Delta_B(z, z') - \Delta_A(Kz, Kz')].$$

Channel $\Phi : A \rightarrow B$ is called *gauge-contravariant*, if

$$\Phi[U_\phi^A \rho (U_\phi^A)^*] = (U_\phi^B)^* \Phi[\rho] U_\phi^B \quad (45)$$

for all input states ρ and all $\phi \in [0, 2\pi]$. For the Gaussian channel (32) with parameters (K, l, μ) this reduces to

$$l(z) \equiv 0, \quad KJ_B + J_A K = 0, \quad \mu(z, z') = \mu(J_B z, J_B z').$$

The relation (32) for gauge-contravariant Gaussian channel takes the form

$$\Phi^*[D_B(\mathbf{z})] = D_A(-\Lambda \mathbf{K} \mathbf{z}) \exp[-\boldsymbol{\mu}(\mathbf{z}, \mathbf{z})], \quad \mathbf{z} \in \mathbf{Z}_B, \quad (46)$$

where Λ is antilinear operator of complex conjugation, $\Lambda^2 = I$, $\Lambda^s = -\Lambda$ in \mathbf{Z}_A such that $\Lambda J_A + J_A \Lambda = 0$, and $\mathbf{K} = -\Lambda K$ is complex linear operator from \mathbf{Z}_B to \mathbf{Z}_A . Here

$$\boldsymbol{\mu} \geq \pm \frac{1}{2} [\mathbf{j}_B + \mathbf{K}^* \mathbf{j}_A \mathbf{K}]. \quad (47)$$

The last condition is obtained by applying Lemma 6 to the Hermitian form

$$\begin{aligned} \beta(z, z') &= \mu(z, z') - \frac{i}{2} [\Delta_B(z, z') - \Delta_A(Kz, Kz')] \\ &= \frac{i}{2} [\Delta_B(z, z') + \Delta_A(\mathbf{K}z, \mathbf{K}z')]. \end{aligned}$$

3.3 Attenuators and amplifiers

In what follows we restrict to channels that are gauge-covariant or contravariant with respect to fixed complex structures. Therefore, to be specific, we consider vectors in \mathbf{Z} as s -dimensional complex column vectors, where the operator J acts as multiplication by i , the corresponding Hermitian inner product is $\mathbf{j}(\mathbf{z}, \mathbf{z}') = \mathbf{z}^* \mathbf{z}'$ and the symplectic form is $\Delta(z, z') = 2\text{Im} \mathbf{z}^* \mathbf{z}'$, where $*$ denotes Hermitian conjugation. The linear operators in \mathbf{Z} commuting with J are represented by complex $s \times s$ -matrices. The gauge group acts in \mathbf{Z} as multiplication by $e^{i\phi}$. Gaussian gauge-invariant states are described by the modified characteristic function

$$\phi(\mathbf{z}) = \text{Tr} \rho D(\mathbf{z}) = \exp(-\mathbf{z}^* \boldsymbol{\alpha} \mathbf{z}), \quad (48)$$

where $\boldsymbol{\alpha}$ is a Hermitian correlation matrix satisfying $\boldsymbol{\alpha} \geq \mathbf{I}/2$ as follows from (37). For the given complex structure, the unique minimal solution of the last inequality is $\frac{1}{2}\mathbf{I}$, to which correspond the vacuum state ρ_0 and the family of coherent states $\{\rho_{\mathbf{z}}; \mathbf{z} \in \mathbf{Z}\}$, such that $\rho_{\mathbf{z}} = D(\mathbf{z})\rho_0 D(\mathbf{z})^*$. One has

$$\text{Tr} \rho_{\mathbf{w}} D(\mathbf{z}) = \exp\left(2i \text{Im} \mathbf{w}^* \mathbf{z} - \frac{1}{2} |\mathbf{z}|^2\right),$$

where $|\mathbf{z}|^2 = \mathbf{z}^* \mathbf{z}$.

Let $\mathbf{Z}_A, \mathbf{Z}_B$ be the input and output spaces of dimensionalities s_A, s_B . We denote by $s_A = \dim \mathbf{Z}_A$, $s_B = \dim \mathbf{Z}_B$ the numbers of modes of the input and output of the channel. The action of a Gaussian gauge-covariant channel (43) can be described as

$$\Phi^*[D_B(\mathbf{z})] = D_A(\mathbf{K}\mathbf{z}) \exp(-\mathbf{z}^* \boldsymbol{\mu} \mathbf{z}), \quad \mathbf{z} \in \mathbf{Z}_B, \quad (49)$$

where \mathbf{K} is complex $s_B \times s_A$ -matrix, $\boldsymbol{\mu}$ is Hermitian $s_B \times s_B$ -matrix satisfying the condition (see [30])

$$\boldsymbol{\mu} \geq \pm \frac{1}{2} (\mathbf{I}_B - \mathbf{K}^* \mathbf{K}), \quad (50)$$

where \mathbf{I}_B is the unit $s_B \times s_B$ -matrix. This follows from (44) by taking into account that the matrix of the form $\mathbf{j}(\mathbf{z}, \mathbf{z}')$ in an orthonormal basis is just the unit matrix \mathbf{I} of the corresponding size. Later we will need the following

Lemma 7 *The map (49) is injective if and only if ⁷ $\mathbf{K}\mathbf{K}^* > \mathbf{0}$ (in which case necessarily $s_B \geq s_A$).*

Proof. Injectivity means that $\Phi[\rho_1] = \Phi[\rho_2]$ implies $\rho_1 = \rho_2$. But $\Phi[\rho_1] = \Phi[\rho_2]$ is equivalent to $\text{Tr}\rho_1\Phi^*[D_B(\mathbf{z})] = \text{Tr}\rho_2\Phi^*[D_B(\mathbf{z})]$, i.e. $\text{Tr}\rho_1 D_A(\mathbf{K}\mathbf{z}) = \text{Tr}\rho_2 D_A(\mathbf{K}\mathbf{z})$ for all $\mathbf{z} \in \mathbf{Z}_B$. By irreducibility of the Weyl system, this property is equivalent to $\text{Ran}\mathbf{K} = \mathbf{Z}_A$, i.e. $\text{Ker}\mathbf{K}^* = \{\mathbf{0}\}$ or $\mathbf{K}\mathbf{K}^* > \mathbf{0}$. ■

The channel (49) is extreme if μ is a minimal solution of the inequality (50). Special cases of the maps (49) are provided by the *attenuator* and *amplifier* channels, characterized by matrix \mathbf{K} fulfilling the inequalities, $\mathbf{K}^*\mathbf{K} \leq \mathbf{I}$ and $\mathbf{K}^*\mathbf{K} \geq \mathbf{I}$ respectively. We are particularly interested in *extreme attenuator* which corresponds to

$$\mathbf{K}^*\mathbf{K} \leq \mathbf{I}_B, \quad \mu = \frac{1}{2}(\mathbf{I}_B - \mathbf{K}^*\mathbf{K}), \quad (51)$$

and *extreme amplifier*

$$\mathbf{K}^*\mathbf{K} \geq \mathbf{I}_B, \quad \mu = \frac{1}{2}(\mathbf{K}^*\mathbf{K} - \mathbf{I}_B). \quad (52)$$

Denoting by $\bar{\mathbf{z}}$ the column vector obtained by taking the complex conjugate of the elements of \mathbf{z} , the action of the Gaussian gauge-contravariant channel (46) is described as

$$\Phi^*[D_B(\mathbf{z})] = D_A(-\overline{\mathbf{K}\mathbf{z}}) \exp(-\mathbf{z}^* \mu \mathbf{z}), \quad (53)$$

where μ is Hermitian matrix satisfying the inequality

$$\mu \geq \frac{1}{2}(\mathbf{I}_B + \mathbf{K}^*\mathbf{K}), \quad (54)$$

which follows from (47). Here $\bar{\mathbf{z}}$ is the column vector consisting of complex conjugates of the components of \mathbf{z} . These maps are extreme if

$$\mu = \frac{1}{2}(\mathbf{I}_B + \mathbf{K}^*\mathbf{K}). \quad (55)$$

The following proposition generalizes to many modes the decomposition of one-mode channels the usefulness of which was emphasized and exploited in the paper [20] (see also [13] on concatenations of one-mode channels):

⁷ For Hermitian matrices M, N , the strict inequality $M > N$ means that $M - N$ is positive definite.

Proposition 8 *Any Gaussian gauge-covariant channel $\Phi : A \rightarrow B$ is a concatenation $\Phi = \Phi_2 \circ \Phi_1$ of extreme attenuator $\Phi_1 : A \rightarrow B$ and extreme amplifier $\Phi_2 : B \rightarrow B$.*

Any Gaussian gauge-contravariant channel $\Phi : A \rightarrow B$ is a concatenation of extreme attenuator $\Phi_1 : A \rightarrow B$ and extreme gauge-contravariant channel $\Phi_2 : B \rightarrow B$.

Proof. The concatenation $\Phi = \Phi_2 \circ \Phi_1$ of Gaussian gauge-covariant channels Φ_1 and Φ_2 obeys the rule:

$$\mathbf{K} = \mathbf{K}_1 \mathbf{K}_2, \quad (56)$$

$$\boldsymbol{\mu} = \mathbf{K}_2^* \boldsymbol{\mu}_1 \mathbf{K}_2 + \boldsymbol{\mu}_2. \quad (57)$$

By inserting relations

$$\boldsymbol{\mu}_1 = \frac{1}{2} (\mathbf{I}_B - \mathbf{K}_1^* \mathbf{K}_1) = \frac{1}{2} (\mathbf{I}_B - |\mathbf{K}_1|^2), \quad \boldsymbol{\mu}_2 = \frac{1}{2} (\mathbf{K}_2^* \mathbf{K}_2 - \mathbf{I}_B) = \frac{1}{2} (|\mathbf{K}_2|^2 - \mathbf{I}_B)$$

into (57) and using (56) we obtain

$$|\mathbf{K}_2|^2 = \mathbf{K}_2^* \mathbf{K}_2 = \boldsymbol{\mu} + \frac{1}{2} (\mathbf{K}^* \mathbf{K} + \mathbf{I}_B) \geq \begin{Bmatrix} \mathbf{I}_B \\ \mathbf{K}^* \mathbf{K} \end{Bmatrix} \quad (58)$$

from the inequality (50). By using operator monotonicity of the square root, we have

$$|\mathbf{K}_2| \geq \mathbf{I}_B, \quad |\mathbf{K}_2| \geq |\mathbf{K}|.$$

The first inequality (58) implies that choosing

$$\mathbf{K}_2 = |\mathbf{K}_2| = \sqrt{\boldsymbol{\mu} + \frac{1}{2} (\mathbf{K}^* \mathbf{K} + \mathbf{I}_B)} \quad (59)$$

and the corresponding $\boldsymbol{\mu}_2 = \frac{1}{2} (|\mathbf{K}_2|^2 - \mathbf{I}_B)$, we obtain extreme amplifier $\Phi_2 : B \rightarrow B$.

Then with

$$\mathbf{K}_1 = \mathbf{K} |\mathbf{K}_2|^{-1} \quad (60)$$

we obtain, taking into account the second inequality in (58) and also Lemma 9 below,

$$\mathbf{K}_1 \mathbf{K}_1^* = \mathbf{K} |\mathbf{K}_2|^{-2} \mathbf{K}^* = \mathbf{K} \left[\boldsymbol{\mu} + \frac{1}{2} (\mathbf{K}^* \mathbf{K} + \mathbf{I}) \right]^{-1} \mathbf{K}^* \leq \mathbf{I}_A, \quad (61)$$

which implies $\mathbf{K}_1^* \mathbf{K}_1 \leq \mathbf{I}_A$, hence \mathbf{K}_1 with the corresponding $\boldsymbol{\mu}_1 = \frac{1}{2} (\mathbf{I}_B - \mathbf{K}_1^* \mathbf{K}_1)$ give the quantum-limited attenuator.

Lemma 9 *Let $\mathbf{M} \geq \mathbf{K}^*\mathbf{K}$, then $\mathbf{KM}^-\mathbf{K}^* \leq \mathbf{I}_A$, where $-$ means (generalized) inverse.*

Proof. By the definition of the generalized inverse,

$$u^*\mathbf{M}^-u = \sup_{v:v \in \text{Ran}\mathbf{M}, v \neq 0} \frac{|u^*v|^2}{v^*\mathbf{M}v}.$$

By inserting \mathbf{K}^*u in place of u and using Cauchy-Schwarz inequality in the nominator of the fraction, we obtain

$$u^*\mathbf{KM}^-\mathbf{K}^*u \leq \sup_{v:v \in \text{Ran}\mathbf{M}, v \neq 0} \frac{u^*u v^*\mathbf{K}^*\mathbf{K}v}{v^*\mathbf{M}v} \leq u^*u.$$

■

In the case of contravariant channel the relations (56), (57) are replaced with

$$\bar{\mathbf{K}} = \mathbf{K}_1\bar{\mathbf{K}}_2, \quad (62)$$

$$\boldsymbol{\mu} = \mathbf{K}_2^*\bar{\boldsymbol{\mu}}_1\mathbf{K}_2 + \boldsymbol{\mu}_2. \quad (63)$$

By substituting

$$\boldsymbol{\mu}_1 = \frac{1}{2}(\mathbf{I} - \mathbf{K}_1^*\mathbf{K}_1), \quad \boldsymbol{\mu}_2 = \frac{1}{2}(\mathbf{K}_2^*\mathbf{K}_2 + \mathbf{I}_B)$$

into (63) and using (54) we obtain

$$|\mathbf{K}_2|^2 = \mathbf{K}_2^*\mathbf{K}_2 = \boldsymbol{\mu} + \frac{1}{2}(\mathbf{K}^*\mathbf{K} - \mathbf{I}_B) \geq \mathbf{K}^*\mathbf{K}. \quad (64)$$

Taking $\mathbf{K}_2 = |\mathbf{K}_2|$, $\boldsymbol{\mu}_2 = \frac{1}{2}(|\mathbf{K}_2|^2 + \mathbf{I}_B)$ gives extreme gauge-contravariant channel $\Phi_2 : B \rightarrow B$. With

$$\bar{\mathbf{K}}_1 = \mathbf{K}|\mathbf{K}_2|^- \quad (65)$$

we obtain, by using Lemma 9,

$$\begin{aligned} \bar{\mathbf{K}}_1\bar{\mathbf{K}}_1^* &= \mathbf{K}(|\mathbf{K}_2|^-)^2\mathbf{K}^* \\ &= \mathbf{K}\left[\boldsymbol{\mu} + \frac{1}{2}(\mathbf{K}^*\mathbf{K} - \mathbf{I}_B)\right]^- \mathbf{K}^* \leq \mathbf{I}_A, \end{aligned} \quad (66)$$

which implies $\mathbf{K}_1\mathbf{K}_1^* \leq \mathbf{I}_A$, with the corresponding $\boldsymbol{\mu}_1$ give the extreme attenuator $\Phi_1 : A \rightarrow B$. ■

Remark 10 In the case of gauge-covariant channel, the equality in (61) shows that $\mathbf{K}\mathbf{K}^* > 0$ implies $\mathbf{K}_1\mathbf{K}_1^* > 0$, while the inequality $\boldsymbol{\mu} > \frac{1}{2}(\mathbf{K}^*\mathbf{K} - \mathbf{I}_B)$ implies $\mathbf{K}_1\mathbf{K}_1^* < \mathbf{I}_A$. In the case of gauge-contravariant channel, the inequality $\boldsymbol{\mu} > \frac{1}{2}(\mathbf{I}_B + \mathbf{K}^*\mathbf{K})$ implies $0 < \mathbf{K}_1\mathbf{K}_1^* < \mathbf{I}_A$ via (66).

Proposition 11 The extreme attenuator with matrix \mathbf{K} and extreme attenuator with matrix $\tilde{\mathbf{K}} = \sqrt{\mathbf{I}_A - \mathbf{K}\mathbf{K}^*}$ are mutually complementary.

The extreme amplifier with matrix \mathbf{K} and gauge-contravariant channel with matrix $\tilde{\mathbf{K}} = \sqrt{\mathbf{K}\mathbf{K}^* - \mathbf{I}_A}$ are mutually complementary.

Proof. For the case of one mode see [13] or [39], Sec. 12.6.1. We sketch the proof for several modes below. Define $\mathbf{Z}_E \simeq \mathbf{Z}_A$, $\mathbf{Z}_D \simeq \mathbf{Z}_B$, so that $\mathbf{Z} = \mathbf{Z}_A \oplus \mathbf{Z}_D \simeq \mathbf{Z}_B \oplus \mathbf{Z}_E$

In the case of attenuator consider the block unitary matrix in \mathbf{Z} :

$$\mathbf{V} = \begin{bmatrix} \mathbf{K} & \sqrt{\mathbf{I}_A - \mathbf{K}\mathbf{K}^*} \\ \sqrt{\mathbf{I}_B - \mathbf{K}^*\mathbf{K}} & -\mathbf{K}^* \end{bmatrix} \quad (67)$$

which defines unitary dynamics U in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_D \simeq \mathcal{H}_B \otimes \mathcal{H}_E$ by the relation $U^*D_{BE}(\mathbf{z}_{BE})U = D_{AD}(\mathbf{V}\mathbf{z}_{BE})$. Here $\mathbf{z}_{BE} = [\mathbf{z}_B \ \mathbf{z}_E]^t$, $D_{BE}(\mathbf{z}_{BE}) = D_B(\mathbf{z}_B) \otimes D_E(\mathbf{z}_E)$, and the unitarity follows from the relation

$$\mathbf{K}\sqrt{\mathbf{I}_B - \mathbf{K}^*\mathbf{K}} = \sqrt{\mathbf{I}_A - \mathbf{K}\mathbf{K}^*}\mathbf{K}. \quad (68)$$

Let $\rho_D = \rho_0$ be the vacuum state, $\rho_A = \rho$ an arbitrary state. Then the formulas (7), (8) define the mutually complementary extreme attenuators as described in the first statement. The proof is obtained by computing the characteristic function of the output states for the channels. For the state of the composite system $\rho_{BE} = U(\rho \otimes \rho_D)U^*$ we have

$$\begin{aligned} \phi_{BE}(\mathbf{z}_{BE}) &= \text{Tr} U(\rho \otimes \rho_D)U^* [D_B(\mathbf{z}_B) \otimes D_E(\mathbf{z}_E)] \\ &= \text{Tr}(\rho \otimes \rho_D)U^* [D_B(\mathbf{z}_B) \otimes D_E(\mathbf{z}_E)] U \\ &= \text{Tr}(\rho \otimes \rho_D) \left[D_A(\mathbf{K}\mathbf{z}_B + \tilde{\mathbf{K}}\mathbf{z}_E) \otimes D_D(\sqrt{\mathbf{I}_B - \mathbf{K}^*\mathbf{K}}\mathbf{z}_B - \mathbf{K}^*\mathbf{z}_E) \right] \\ &= \phi_A(\mathbf{K}\mathbf{z}_B + \tilde{\mathbf{K}}\mathbf{z}_E) \exp \left[-\frac{1}{2} |\sqrt{\mathbf{I}_B - \mathbf{K}^*\mathbf{K}}\mathbf{z}_B - \mathbf{K}^*\mathbf{z}_E|^2 \right]. \end{aligned} \quad (69)$$

By setting $\mathbf{z}_E = 0$ or $\mathbf{z}_B = 0$ we obtain

$$\begin{aligned} \phi_B(\mathbf{z}_B) &= \phi_A(\mathbf{K}\mathbf{z}_B) \exp \left[-\frac{1}{2} \mathbf{z}_B^* (\mathbf{I}_B - \mathbf{K}^*\mathbf{K}) \mathbf{z}_B \right], \\ \phi_E(\mathbf{z}_E) &= \phi_A(\tilde{\mathbf{K}}\mathbf{z}_E) \exp \left[-\frac{1}{2} \mathbf{z}_E^* \mathbf{K}\mathbf{K}^* \mathbf{z}_E \right] \end{aligned}$$

as required.

In the case of amplifier, set

$$\mathbf{V} = \begin{bmatrix} \mathbf{K} & -\sqrt{\mathbf{K}\mathbf{K}^* - \mathbf{I}_A}\Lambda \\ -\Lambda\sqrt{\mathbf{K}^*\mathbf{K} - \mathbf{I}_B} & \Lambda\mathbf{K}^*\Lambda \end{bmatrix},$$

where Λ is the operator of complex conjugation, anticommuting with multiplication by i . By using the property $\Delta(\Lambda z, \Lambda z') = -\Delta(z, z')$, we obtain that \mathbf{V} corresponds to a symplectic transformation in \mathbf{Z} generating unitary dynamics U in \mathcal{H} . Let again ρ_0 be the vacuum state of the environment. Then the formulas (7), (8) define the mutually complementary channels as described in the second statement of the Proposition, and the proof is similar.

To show that \mathbf{V} is a symplectic transformation, introduce the matrices

$$\Theta = \begin{bmatrix} \mathbf{I}_B & 0 \\ 0 & -\Lambda \end{bmatrix}, \quad \mathbf{V}_1 = \begin{bmatrix} \mathbf{K} & \sqrt{\mathbf{K}\mathbf{K}^* - \mathbf{I}_A} \\ \sqrt{\mathbf{K}^*\mathbf{K} - \mathbf{I}_B} & \mathbf{K}^* \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \mathbf{I}_B & 0 \\ 0 & -\mathbf{I}_A \end{bmatrix}.$$

Notice that $\mathbf{V} = \Theta\mathbf{V}_1\Theta$, and $\mathbf{V}_1^*\Sigma\mathbf{V}_1 = \Sigma$, which means that \mathbf{V}_1 preserves the indefinite Hermitian form $\sigma(z_{BE}, z'_{BE}) = z_B^*z'_B - z_E^*z'_E$. By taking into account

$$\Delta_{BE}(\Theta z_{BE}, \Theta z'_{BE}) = \Delta_B(z_B, z'_B) - \Delta_E(z_E, z'_E) = \text{Im}\sigma(z_{BE}, z'_{BE}),$$

we obtain

$$\Delta_{BE}(\mathbf{V}z_{BE}, \mathbf{V}z'_{BE}) = \text{Im}\sigma(\mathbf{V}_1\Theta z_{BE}, \mathbf{V}_1\Theta z'_{BE}) = \text{Im}\sigma(\Theta z_{BE}, \Theta z'_{BE}) = \Delta_{BE}(z_{BE}, z'_{BE}),$$

as required. ■

Again, later we will need the following

Lemma 12 *Let $\Phi_1 : A \rightarrow B$ be an extreme attenuator with $\mathbf{0} < \mathbf{K}_1\mathbf{K}_1^* < \mathbf{I}_A$, then $\Phi_1[P_\psi] = P_{\psi'}$ (a pure state) if and only if P_ψ is a coherent state.*

Proof. According to Proposition 11, the complementary channel $\tilde{\Phi}_1$ is an extreme attenuator with the matrix $\tilde{\mathbf{K}} = \sqrt{\mathbf{I}_A - \mathbf{K}_1\mathbf{K}_1^*}$, such that $\mathbf{0} < \tilde{\mathbf{K}} < \mathbf{I}_A$. Its output is also pure, $\Phi_1[P_\psi] = P_{\psi'_E}$, as the outputs of complementary channels have identical nonzero spectra by Lemma 2. Thus

$$U(\psi \otimes \psi_0) = \psi' \otimes \psi'_E,$$

where $\psi_0 \in \mathcal{H}_D$ is the vacuum vector and U is the unitary operator in \mathcal{H} implementing the symplectic transformation corresponding to the unitary (67) in $\mathbf{Z}_A \oplus \mathbf{Z}_D \simeq \mathbf{Z}_B \oplus \mathbf{Z}_E$, with $\mathbf{Z}_D \simeq \mathbf{Z}_B$, $\mathbf{Z}_E \simeq \mathbf{Z}_A$. Denoting by

$$\phi(\mathbf{z}) = \text{Tr} P_\psi D_A(\mathbf{z}), \quad \phi'(\mathbf{z}_B) = \text{Tr} P_{\psi'} D_B(\mathbf{z}_B), \quad \phi_E(\mathbf{z}_E) = \text{Tr} P_{\psi'_E} D_E(\mathbf{z}_E)$$

the quantum characteristic functions and using the relation (69), we have the functional equation

$$\phi'(\mathbf{z}_B) \phi_E(\mathbf{z}_E) = \phi(\mathbf{K}_1 \mathbf{z}_B + \tilde{\mathbf{K}} \mathbf{z}_E) \exp \left[-\frac{1}{2} |\sqrt{\mathbf{I}_B - \mathbf{K}_1^* \mathbf{K}_1} \mathbf{z}_B - \mathbf{K}_1^* \mathbf{z}_E|^2 \right]. \quad (70)$$

By letting $\mathbf{z}_E = 0$, respectively $\mathbf{z} = 0$, we obtain

$$\begin{aligned} \phi'(\mathbf{z}_B) &= \phi(\mathbf{K}_1 \mathbf{z}_B) \exp \left[-\frac{1}{2} |\sqrt{\mathbf{I}_B - \mathbf{K}_1^* \mathbf{K}_1} \mathbf{z}_B|^2 \right], \\ \phi_E(\mathbf{z}_E) &= \phi(\tilde{\mathbf{K}} \mathbf{z}_E) \exp \left[-\frac{1}{2} |\mathbf{K}_1^* \mathbf{z}_E|^2 \right], \end{aligned}$$

thus, after the change of variables $\mathbf{z} = \mathbf{K}_1 \mathbf{z}_B$, $\mathbf{z}' = \tilde{\mathbf{K}} \mathbf{z}_E$, and using (68), the equation (70) reduces to

$$\phi(\mathbf{z}) \phi(\mathbf{z}') = \phi(\mathbf{z} + \mathbf{z}') \exp [\text{Re } \mathbf{z}^* \mathbf{z}'].$$

The condition of the Lemma ensures that $\text{Ran } \mathbf{K}_1 = \text{Ran } \tilde{\mathbf{K}} = \mathbf{Z}_A$. Substituting $\omega(\mathbf{z}) = \phi(\mathbf{z}) \exp [\frac{1}{2} |\mathbf{z}|^2]$, this becomes

$$\omega(\mathbf{z}) \omega(\mathbf{z}') = \omega(\mathbf{z} + \mathbf{z}') \quad (71)$$

for all $\mathbf{z}, \mathbf{z}' \in \mathbf{Z}_A$. The function $\omega(\mathbf{z})$, as well as the characteristic function $\phi(\mathbf{z})$, is continuous and satisfies $\omega(-\mathbf{z}) = \overline{\omega(\mathbf{z})}$. The only solution of (71) satisfying these conditions is the exponent $\omega(\mathbf{z}) = \exp [i \text{Im } \mathbf{w}^* \mathbf{z}]$ for some complex \mathbf{w} . Thus

$$\phi(\mathbf{z}) = \exp \left[i \text{Im } \mathbf{w}^* \mathbf{z} - \frac{1}{2} |\mathbf{z}|^2 \right]$$

is the characteristic function of the coherent state $\rho_{\mathbf{w}/2}$. ■

3.4 Gaussian optimizers

The following basic result for one mode was obtained in [53]. Here we present a complete proof in the multimode case, a sketch of which was given in [22].

Theorem 13 (i) *Let Φ be a gauge covariant or contravariant channel and let f be a real concave function on $[0, 1]$, such that $f(0) = 0$, then*

$$\text{Tr} f(\Phi[\rho]) \geq \text{Tr} f(\Phi[\rho_{\mathbf{w}}]) = \text{Tr} f(\Phi[\rho_0]) \quad (72)$$

for all states ρ and any coherent state $\rho_{\mathbf{w}}$ (the value on the right is the same for all coherent states by the unitary covariance property of a Gaussian channel (34)).

(ii) *Let f be strictly concave, then equality in (72) is attained only if ρ is a coherent state in the following cases:*

- a) $s_B = s_A$ and Φ is an extreme amplifier with $\boldsymbol{\mu} = \frac{1}{2}(\mathbf{K}^*\mathbf{K} - \mathbf{I}_B) > 0$;*
- b) $s_B \geq s_A$, the channel Φ is gauge-covariant with $\mathbf{K}\mathbf{K}^* > 0$ and*

$$\boldsymbol{\mu} > \frac{1}{2}(\mathbf{K}^*\mathbf{K} - \mathbf{I}_B); \quad (73)$$

- c) $s_B \geq s_A$, the channel Φ is gauge-contravariant with $\mathbf{K}\mathbf{K}^* > 0$ and $\boldsymbol{\mu} > \frac{1}{2}(\mathbf{I}_B + \mathbf{K}^*\mathbf{K})$.*

Proof. (i) We first prove the inequality (72) for strictly concave f . Then the inequality for arbitrary concave f follows by the monotone approximation $f(x) = \lim_{\varepsilon \downarrow 0} f_\varepsilon(x)$, since $f_\varepsilon(x) = f(x) - \varepsilon x^2$ are strictly concave. Also, by concavity, it is sufficient to prove (72) only for $\rho = P_\psi$.

By Proposition 8, $\Phi = \Phi_2 \circ \Phi_1$ where $\Phi_1 : A \rightarrow B$ is an extreme attenuator and $\Phi_2 : B \rightarrow B$ is either extreme amplifier or extreme gauge-contravariant channel. Any extreme attenuator maps vacuum state into vacuum. Indeed,

$$\begin{aligned} \text{Tr} \Phi_1[\rho_0] D_B(\mathbf{z}) &= \text{Tr} \rho_0 \Phi_1^*[D_B(\mathbf{z})] \\ &= \text{Tr} \rho_0 D_A(\mathbf{K}\mathbf{z}) \exp\left(-\frac{1}{2}\mathbf{z}^*(\mathbf{I}_B - \mathbf{K}^*\mathbf{K})\mathbf{z}\right) \\ &= \exp\left(-\frac{1}{2}|\mathbf{z}|^2\right) = \text{Tr} \rho_0 D_B(\mathbf{z}). \end{aligned}$$

Therefore $\text{Tr}f(\Phi[\rho_0]) = \text{Tr}f(\Phi_2[\rho_0])$. Then it is sufficient to prove (72) for all extreme amplifiers and all extreme gauge-contravariant channels Φ_2 . Indeed, assume that we have proved

$$\text{Tr}f(\Phi_2[P_\psi]) \geq \text{Tr}f(\Phi_2[\rho_0]). \quad (74)$$

for any state vector ψ . Consider the spectral decomposition $\Phi_1[P_\psi] = \sum_j p_j P_{\phi_j}$, where $p_j > 0$, then

$$\text{Tr}f(\Phi[P_\psi]) = \text{Tr}f(\Phi_2[\Phi_1[P_\psi]]) \quad (75)$$

$$\geq \sum_j p_j \text{Tr}f(\Phi_2[P_{\phi_j}]) \quad (76)$$

$$\geq \text{Tr}f(\Phi_2[\rho_0]) \quad (77)$$

$$= \text{Tr}f(\Phi_2[\Phi_1[\rho_0]]) = \text{Tr}f(\Phi[\rho_0]). \quad (78)$$

Then, according to the second statement of Proposition 11 and Lemma 2

$$\text{Tr}f(\Phi_2[P_\psi]) = \text{Tr}f(\tilde{\Phi}_2[P_\psi]),$$

where Φ_2 is an extreme amplifier and $\tilde{\Phi}_2$ is an extreme gauge-contravariant channel. Thus it is sufficient to prove (74) only for an extreme amplifier $\Phi_2 : B \rightarrow B$, with Hermitian matrix $\mathbf{K}_2 \geq \mathbf{I}_B$.

The following result is based on a key observation by Giovannetti.

Lemma 14 *For an extreme amplifier $\Phi_2 : B \rightarrow B$, with matrix $\mathbf{K}_2 \geq \mathbf{I}_B$, there is an extreme attenuator Φ'_1 such that for all $\psi \in \mathcal{H}_B$*

$$\Phi_2(P_\psi) \sim (\Phi_2 \circ \Phi'_1)(P_\psi). \quad (79)$$

Proof. By Proposition 11 and Lemma 2 $\Phi_2(P_\psi) \sim \tilde{\Phi}_2(P_\psi)$ for all $\psi \in \mathcal{H}_B$, where $\tilde{\Phi}_2$ is extreme contravariant channel with the matrix $\tilde{\mathbf{K}} = \sqrt{\mathbf{K}_2^2 - \mathbf{I}_B}$.

Define the transposition map $\mathcal{T} : B \rightarrow B$ by the relation $\mathcal{T}[D(\mathbf{z})] = D(-\bar{\mathbf{z}})$. The concatenation $\Phi = \mathcal{T} \circ \tilde{\Phi}_2$ is a covariant Gaussian channel:

$$\Phi^*[D(\mathbf{z})] = \tilde{\Phi}_2^* \circ \mathcal{T}[D(\mathbf{z})] = D(\sqrt{\mathbf{K}_2^2 - \mathbf{I}_B} \mathbf{z}) \exp\left(-\frac{1}{2} \mathbf{z}^* \mathbf{K}_2^2 \mathbf{z}\right).$$

Applying decomposition from Proposition 8, namely the relation (58), gives $\Phi = \Phi_2 \circ \Phi'_1$, where Φ_2 is the original amplifier, and $\Phi'_1 : B \rightarrow B$ is another

extreme attenuator with matrix $\mathbf{K}_1 = \sqrt{\mathbf{I}_B - \mathbf{K}_2^{-2}}$. This implies the relation (79). ■

Lemma 14 and Lemma 4 imply

$$\text{Tr} f(\Phi_2(P_\psi)) = \text{Tr} f((\Phi_2 \circ \Phi'_1)(P_\psi)). \quad (80)$$

Again, consider the spectral decomposition of the density operator

$$\Phi'_1(P_\psi) = \sum_j p'_j P_{\psi_j}, \quad p'_j > 0.$$

By concavity,

$$\text{Tr} f((\Phi_2 \circ \Phi'_1)(P_\psi)) \geq \sum_j p'_j \text{Tr} f(\Phi_2[P_{\psi_j}]). \quad (81)$$

Since f is assumed strictly concave, then $\rho \rightarrow \text{Tr} f(\Phi_2[\rho])$ is strictly concave [11]. Assuming that P_ψ is a minimizer for the functional (80), we conclude that $\Phi_2[P_{\psi_j}]$ must all coincide, otherwise the above inequality would be strict, contradicting the assumption. From Lemma 7 it follows that $P_{\psi_j} = P_{\psi'}$ for all j and for some $\psi' \in \mathcal{H}_B$, hence, assuming that P_ψ is a minimizer, the output $\Phi_1[P_\psi] = P_{\psi'}$ is a pure state.

Since $\mathbf{K}_1 = \sqrt{\mathbf{I}_B - \mathbf{K}_2^{-2}}$, the condition of Lemma 12 is fulfilled if $\mathbf{K}_2 > \mathbf{I}_B$. In this case, if P_ψ is a minimizer, the Lemma implies that P_ψ is a coherent state. Thus we obtain the inequality (74) for the amplifier Φ_2 with $\mathbf{K}_2 > \mathbf{I}_B$ and strictly concave f . In this way we also obtain the case a) of the “only if” statement (ii).

In the case of amplifier Φ_2 with $\mathbf{K}_2 \geq \mathbf{I}_B$, we can take any sequence $\mathbf{K}_2^{(n)} > \mathbf{I}_B$, $\mathbf{K}_2^{(n)} \rightarrow \mathbf{K}_2$, and the corresponding amplifiers $\Phi_2^{(n)}$. Then $\text{Tr} f(\Phi_2^{(n)}[\rho]) \rightarrow \text{Tr} f(\Phi_2[\rho])$ for any concave polygonal function f on $[0, 1]$, such that $f(0) = 0$, and any $\rho \in \mathfrak{S}(\mathcal{H}_A)$. This follows from the fact that any such function is Lipschitz, $|f(x) - f(y)| \leq \varkappa|x - y|$, and $\left\| \Phi_2^{(n)}[\rho] - \Phi_2[\rho] \right\|_1 \rightarrow 0$. It follows that (74) holds for all extreme amplifiers Φ_2 in the case of polygonal concave functions f . For arbitrary concave f on $[0, 1]$ there is a monotonously nondecreasing sequence of concave polygonal functions f_m converging to f pointwise. Passing to the limit $m \rightarrow \infty$ gives the inequality (74) for arbitrary extreme amplifier, and hence, (72) holds for arbitrary Gaussian gauge-covariant or contravariant channels.

(ii) The “only if” statement in the cases b), c) are obtained from the decomposition $\Phi = \Phi_2 \circ \Phi_1$ and the relations (75)-(77) by applying argument similar to the case of extreme amplifier. Notice that the conditions on the channel Φ imply that in the decomposition $\Phi = \Phi_2 \circ \Phi_1$ the attenuator Φ_1 is defined by the matrix \mathbf{K}_1 such that $0 < \mathbf{K}_1 \mathbf{K}_1^* < \mathbf{I}_A$ (see Remark 10). Applying the argument involving the relations (80)-(81) with strictly concave f to the relations (75)-(78), we obtain that for any pure minimizer P_ψ of $\text{Tr} f(\Phi[P_\psi])$ the output of the extremal attenuator $\Phi_1[P_\psi]$ is necessarily a pure state. Applying Lemma 18 to the attenuator Φ_1 we conclude that P_ψ is necessarily a coherent state. ■

3.5 Explicit formulas and additivity

Proposition 15 *For any $p > 1$ and any Gaussian gauge-covariant or contravariant channel Φ*

$$\|\Phi\|_{1 \rightarrow p} = (\text{Tr} \Phi[\rho_0]^p)^{1/p}, \quad (82)$$

$$\check{R}_p(\Phi) = R_p(\Phi[\rho_0]), \quad (83)$$

$$\check{H}(\Phi) = H(\Phi[\rho_0]), \quad (84)$$

where ρ_0 is the vacuum state.

The multiplicativity property (14) holds for any two Gaussian gauge-covariant (contravariant) channels Φ_1 and Φ_2 , as well as the additivity of the minimal Rényi entropy (15) and of the minimal von Neumann entropy (16).

Proof. The first statement follows from Theorem 13 by taking $f(x) = -x^p$, $f(x) = -x \log x$.

If Φ_1 and Φ_2 are both gauge-covariant (contravariant), then their tensor product $\Phi_1 \otimes \Phi_2$ shares this property. The second statement then follows from the expressions (82) - (84) and the product property of the vacuum state $\rho_0 = \rho_0^{(1)} \otimes \rho_0^{(2)}$, which follows from the definition. ■

From the definitions of gauge-co/contravariant channels (49), (53), it follows that the state $\Phi[\rho_0]$ is gauge-invariant Gaussian with the correlation matrix $\boldsymbol{\mu} + \mathbf{K}^* \mathbf{K} / 2$. The spectrum of $\Phi[\rho_0]$ is computed explicitly leading to the expressions [41]

$$\|\Phi\|_{1 \rightarrow p} = [\det [(\boldsymbol{\mu} + \mathbf{K}^* \mathbf{K} / 2 + \mathbf{I}_B / 2)^p - (\boldsymbol{\mu} + \mathbf{K}^* \mathbf{K} / 2 - \mathbf{I}_B / 2)^p]]^{-1/p}$$

and

$$\check{H}(\Phi) = \text{tr } g(\boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B)/2), \quad (85)$$

where $g(x) = (x+1)\log(x+1) - x\log x$ and tr denotes trace of operators in \mathbf{Z} . In the last case we used the formula for the entropy of Gaussian state (48) [40]:

$$H(\rho) = \text{tr } g(\boldsymbol{\alpha} - \mathbf{I}/2).$$

We now turn to the classical capacity of the channel Φ . In infinite dimensions, there are two novel features as compared to the situation described in Sec. 2.4. First, one has to extend the notion of ensemble to embrace continual families of states. We call *generalized ensemble* an arbitrary Borel probability measure π on $\mathfrak{S}(\mathcal{H}_A)$. The *average state* of the generalized ensemble π is defined as the barycenter of the probability measure

$$\bar{\rho}_\pi = \int_{\mathfrak{S}(\mathcal{H}_A)} \rho \pi(d\rho).$$

The conventional ensembles correspond to finitely supported measures.

Second, one has to consider the input constraints to avoid infinite values of the capacities. Let F be a positive selfadjoint operator in \mathcal{H}_A , which usually represents energy in the system A . We consider the input states with constrained energy: $\text{Tr} \rho F \leq E$, where E is a fixed positive constant. Since the operator F is usually unbounded, care should be taken in defining the trace; we put $\text{Tr} \rho F = \int_0^\infty \lambda dm_\rho(\lambda)$, where $m_\rho(\lambda) = \text{Tr} \rho E(\lambda)$, and $E(\lambda)$ is the spectral function of the selfadjoint operator F . Then the constrained χ -capacity is given by the following generalization of the expression (17):

$$C_\chi(\Phi, F, E) = \sup_{\pi: \text{Tr} \bar{\rho}_\pi F \leq E} \chi(\pi), \quad (86)$$

where

$$\chi(\pi) = H(\Phi[\bar{\rho}_\pi]) - \int_{\mathfrak{S}(\mathcal{H}_A)} H(\Phi[\rho]) \pi(d\rho) \quad (87)$$

To ensure that this expression is defined correctly, certain additional conditions upon the channel Φ and the constraint operator F should be imposed (see [39], Sec. 11.5), which however are always fulfilled in the Gaussian case we consider below.

Denote $F^{(n)} = F \otimes I \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes F$, then the *constrained classical capacity* is given by the expression

$$C(\Phi, F, E) = \lim_{n \rightarrow \infty} \frac{1}{n} C_\chi(\Phi^{\otimes n}, F^{(n)}, nE). \quad (88)$$

Now let Φ be a Gaussian gauge-covariant channel, and consider gauge-invariant oscillator energy operator $F = \sum_{j,k=1}^{s_A} \epsilon_{jk} a_j^* a_k$, where $\epsilon = [\epsilon_{jk}]$ is a Hermitian positive definite matrix, $a_j = \frac{1}{\sqrt{2}}(q_j + ip_j)$ – the annihilation operator for j -th mode. For any state ρ satisfying $\text{Tr} \rho F < \infty$, the first moments $\text{Tr} \rho a_j$ and the second moments $\text{Tr} \rho a_j^* a_k$, $\text{Tr} \rho a_j a_k$ are well defined. For gauge-invariant state $\text{Tr} \rho a_j = 0$ and $\text{Tr} \rho a_j a_k = 0$. For a Gaussian gauge-invariant state (48)

$$\alpha - \mathbf{I}/2 = [\text{Tr} \bar{\rho}_\pi a_j^* a_k]_{j,k=1,\dots,s},$$

see e.g. [33].

Proposition 16 *The constrained classical capacity of the Gaussian gauge-covariant channel Φ is*

$$\begin{aligned} C(\Phi; F, E) &= C_\chi(\Phi; F, E) \\ &= \max_{\nu: \text{tr} \nu \epsilon \leq E} \text{tr} g(\mathbf{K}^* \nu \mathbf{K} + \boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B) / 2) - \text{tr} g(\boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B) / 2). \end{aligned} \quad (89)$$

The optimal ensemble π which attains the supremum in (86) consists of coherent states $\rho_{\mathbf{z}} = D_A(\mathbf{z}) \rho_0 D_A(\mathbf{z})^*$, $\mathbf{z} \in \mathbf{Z}_A$ distributed with gauge-invariant Gaussian probability distribution $Q_\nu(d^{2s} z)$ on \mathbf{Z}_A having zero mean and the correlation matrix ν which solves the maximization problem in (89).

Proof. Consider a Gaussian ensemble π_ν consisting of coherent states $\rho_{\mathbf{z}} = D_A(\mathbf{z}) \rho_0 D_A(\mathbf{z})^*$, $\mathbf{z} \in \mathbf{Z}_A$, with gauge-invariant Gaussian probability distribution $Q_\nu(d^{2s} z)$ on \mathbf{Z}_A having zero mean and some correlation matrix ν . It is defined by the classical characteristic function

$$\int_{\mathbf{Z}_A} \exp(2i \text{Im} \mathbf{w}^* \mathbf{z}) Q_\nu(d^{2s} w) = \exp(-\mathbf{z}^* \nu \mathbf{z}).$$

By using the covariance property (34) of Gaussian channel, we have

$$H(\Phi[\rho_{\mathbf{z}}]) = H(\Phi[D_A(\mathbf{z}) \rho_0 D_A(\mathbf{z})^*]) = H(\Phi[\rho_0]) = \text{tr} g(\boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B) / 2),$$

which does not depend on \mathbf{z} , and hence it gives the value of the integral term in (87). Integration of the characteristic functions of coherent states gives

$$\mathrm{Tr} \bar{\rho}_{\pi_{\nu}} D_A(\mathbf{z}) = \exp(-\mathbf{z}^* (\boldsymbol{\nu} + \mathbf{I}_A/2) \mathbf{z}).$$

Then $\boldsymbol{\nu} = [\mathrm{Tr} \bar{\rho}_{\pi} a_j^* a_k]_{j,k=1,\dots,s_A}$ and $\mathrm{Tr} \bar{\rho}_{\pi_{\nu}} F = \sum_{j,k=1}^s \epsilon_{jk} \mathrm{Tr} \bar{\rho}_{\pi_{\nu}} a_j^* a_k = \mathrm{tr} \boldsymbol{\nu} \boldsymbol{\epsilon}$. The state $\Phi[\bar{\rho}_{\pi_{\nu}}]$ is gauge-invariant Gaussian with the correlation matrix $\mathbf{K}^* (\boldsymbol{\nu} + \mathbf{I}_A/2) \mathbf{K} + \boldsymbol{\mu}$, hence it has the entropy $\mathrm{tr} g(\mathbf{K}^* \boldsymbol{\nu} \mathbf{K} + \boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B)/2)$. Thus for the Gaussian ensemble π_{ν}

$$\chi(\pi_{\nu}) = \mathrm{tr} g(\mathbf{K}^* \boldsymbol{\nu} \mathbf{K} + \boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B)/2) - \mathrm{tr} g(\boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B)/2). \quad (90)$$

Summarizing, we need to show

$$C(\Phi; F, E) = C_{\chi}(\Phi; F, E) = \sup_{\boldsymbol{\nu}: \mathrm{tr} \boldsymbol{\nu} \boldsymbol{\epsilon} \leq E} \chi(\pi_{\nu}). \quad (91)$$

Let us denote by \mathcal{G} the set of Gaussian gauge-invariant states in \mathcal{H}_A .

Lemma 17

$$\max_{\rho^{(n)}: \mathrm{Tr} \rho^{(n)} F^{(n)} \leq nE} H(\Phi^{\otimes n}[\rho^{(n)}]) \leq n \max_{\rho: \rho \in \mathcal{G}, \mathrm{Tr} \rho F \leq E} H(\Phi[\rho]). \quad (92)$$

Proof. We first prove that

$$\sup_{\rho^{(n)}: \mathrm{Tr} \rho^{(n)} F^{(n)} \leq nE} H(\Phi^{\otimes n}[\rho^{(n)}]) \leq n \sup_{\rho: \mathrm{Tr} \rho F \leq E} H(\Phi[\rho]). \quad (93)$$

Indeed, denoting by ρ_j the partial state of $\rho^{(n)}$ in the j -th tensor factor of $\mathcal{H}_A^{\otimes n}$ and letting $\bar{\rho} = \frac{1}{n} \sum_{j=1}^n \rho_j$, we have

$$H(\Phi^{\otimes n}[\rho^{(n)}]) \leq \sum_{j=1}^n H(\Phi[\rho_j]) \leq n H(\Phi[\bar{\rho}]),$$

where in the first inequality we used subadditivity of the quantum entropy, while in the second – its concavity. Moreover, $\mathrm{Tr} \bar{\rho} F = \frac{1}{n} \mathrm{Tr} \rho^{(n)} F^{(n)} \leq E$, hence (93) follows.

Using gauge covariance of the channel Φ , we can then reduce maximization in the right hand side of (93) to gauge-invariant states. Indeed, for a given state ρ , satisfying the constraint $\mathrm{Tr} \rho F \leq E$ the averaging

$$\rho_{av} = \frac{1}{2\pi} \int_0^{2\pi} U_{\varphi} \rho U_{\varphi}^* d\varphi$$

also satisfies the constraint, while $H(\Phi[\rho]) \leq H(\Phi[\rho_{av}])$ by concavity of the entropy.

Finally, we use the maximum entropy principle which says that among states with fixed second moments the Gaussian state has maximal entropy (see e.g. [39], Lemma 12.25). This proves (92). ■

We have

$$\max_{\rho: \rho \in \mathcal{G}, \text{Tr} \rho F \leq E} H(\Phi[\rho]) = \max_{\boldsymbol{\nu}: \text{tr} \boldsymbol{\nu} \boldsymbol{\epsilon} \leq E} \text{tr} g(\mathbf{K}^* \boldsymbol{\nu} \mathbf{K} + \boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B)/2). \quad (94)$$

Now let $\boldsymbol{\nu}$ be the solution of the maximization problem in the righthand side. To prove (89) observe that

$$\begin{aligned} n\chi(\pi_{\boldsymbol{\nu}}) &\leq nC_{\chi}(\Phi, F, E) \leq C_{\chi}(\Phi^{\otimes n}, F^{(n)}, nE) \\ &\leq \max_{\rho^{(n)}: \text{Tr} \rho^{(n)} F^{(n)} \leq nE} H(\Phi^{\otimes n}[\rho^{(n)}]) - \min_{\rho^{(n)}} H(\Phi^{\otimes n}[\rho^{(n)}]). \end{aligned}$$

By using Lemma 17 and Proposition 15 we see that this is less than or equal to

$$n \left[\max_{\rho: \rho \in \mathcal{G}, \text{Tr} \rho F \leq E} H(\Phi[\rho]) - H(\Phi[\rho_0]) \right] = n\chi(\pi_{\boldsymbol{\nu}}),$$

where the equality follows from (94) and (90).

Thus $C_{\chi}(\Phi^{\otimes n}, F^{(n)}, nE) = nC_{\chi}(\Phi, F, E)$ and hence the constrained classical capacity (88) of the Gaussian gauge-covariant channel is given by the expression (89). ■

Similar argument applies to Gaussian gauge-contravariant channel (53), giving the expression (89) with $\boldsymbol{\epsilon}$ replaced by $\bar{\boldsymbol{\epsilon}}$. Indeed, in this case the state $\Phi[\bar{\rho}_{\pi_{\boldsymbol{\nu}}}]$ is gauge-invariant Gaussian with the characteristic function

$$\begin{aligned} \text{Tr} \Phi[\bar{\rho}_{\pi_{\boldsymbol{\nu}}}] D(\mathbf{z}) &= \exp(-(\overline{\mathbf{K}\mathbf{z}})^* (\boldsymbol{\nu} + \mathbf{I}_A/2) \overline{\mathbf{K}\mathbf{z}} - \mathbf{z}^* \boldsymbol{\mu} \mathbf{z}) \\ &= \exp(-(\mathbf{K}\mathbf{z})^* (\bar{\boldsymbol{\nu}} + \mathbf{I}_A/2) \mathbf{K}\mathbf{z} - \mathbf{z}^* \boldsymbol{\mu} \mathbf{z}), \end{aligned}$$

with the correlation matrix $\mathbf{K}^* (\bar{\boldsymbol{\nu}} + \mathbf{I}_A/2) \mathbf{K} + \boldsymbol{\mu}$. On the other hand, $\text{tr} \bar{\boldsymbol{\nu}} \bar{\boldsymbol{\epsilon}} = \text{tr} \bar{\boldsymbol{\nu}} \bar{\boldsymbol{\epsilon}}$, so that redefining $\bar{\boldsymbol{\nu}}$ as $\boldsymbol{\nu}$, we get the statement.

The maximization in (89) is a finite-dimensional optimization problem which is a quantum analog of “water-filling” problem in classical information theory, see e.g. [15, 40]. It can be solved explicitly only in some special cases, e.g. when $\mathbf{K}, \boldsymbol{\mu}, \boldsymbol{\epsilon}$ commute, and it is a subject of separate study.

3.6 The case of quantum-classical Gaussian channel

Consider affine map which transforms quantum states $\rho \in \mathfrak{S}(\mathcal{H})$ into probability densities on \mathbf{Z}

$$\rho \rightarrow p_\rho(\mathbf{z}) = \text{Tr} \rho D(\mathbf{z}) \rho_0 D(\mathbf{z})^*, \quad (95)$$

where $D(\mathbf{z})$ are the displacement operators, ρ_0 is the vacuum state with the quantum characteristic function

$$\phi_0(\mathbf{z}) \equiv \text{Tr} \rho_0 D(\mathbf{z}) = \exp \left(-\frac{1}{2} \mathbf{z}^* \mathbf{z} \right),$$

The function $p_\rho(\mathbf{z})$ is bounded by 1 and is indeed a continuous probability density, the normalization follows from the resolution of the identity

$$\int_{\mathbf{Z}} D(\mathbf{z}) \rho_0 D(\mathbf{z})^* \frac{d^{2s} \mathbf{z}}{\pi^s} = I.$$

Proposition 18 *Let f be a concave function on $[0, 1]$, such that $f(0) = 0$, then for arbitrary state ρ*

$$\int_{\mathbf{Z}} f(p_\rho(\mathbf{z})) \frac{d^{2s} \mathbf{z}}{\pi^s} \geq \int_{\mathbf{Z}} f(p_{\rho_{\mathbf{w}}}(\mathbf{z})) \frac{d^{2s} \mathbf{z}}{\pi^s}. \quad (96)$$

Proof. For any $c > 0$ consider the channel Φ_c defined by the relation

$$\Phi_c[\rho] = \int \frac{d^{2s} \mathbf{z}}{\pi^s c^{2s}} \text{Tr}[\rho D(c^{-1} \mathbf{z}) \rho_0 D^*(c^{-1} \mathbf{z})] \rho_{\mathbf{z}}. \quad (97)$$

The map (97) is a Gaussian gauge-covariant channel such that

$$\Phi_c^*[D(\mathbf{z})] = D(c\mathbf{z}) \exp \left[-\frac{(c^2 + 1)}{2} |\mathbf{z}|^2 \right],$$

cf. [23]. Therefore by Theorem 13,

$$\text{Tr} f(\Phi_c[\rho]) \geq \text{Tr} f(\Phi_c[\rho_{\mathbf{w}}]) \quad (98)$$

for all states ρ and any coherent state $\rho_{\mathbf{w}}$. We will prove the Proposition 18 by taking the limit $c \rightarrow \infty$.

In the proof we also use a simple generalization of the Berezin-Lieb inequalities [9]:

$$\int_{\mathbf{z}} f(\underline{p}(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s} \leq \text{Tr} f(\sigma) \leq \int_{\mathbf{z}} f(\bar{p}(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s}, \quad (99)$$

valid for any quantum state admitting the representation

$$\sigma = \int_{\mathbf{z}} \underline{p}(\mathbf{z}) \rho_{\mathbf{z}} \frac{d^{2s}\mathbf{z}}{\pi^s}$$

with a probability density $\underline{p}(\mathbf{z})$. In the right side of (99) $\bar{p}(\mathbf{z}) = \text{Tr} \sigma \rho_{\mathbf{z}}$. In the inequalities (99) one has to assume that f is defined on $[0, \infty)$ (in fact, $\underline{p}(\mathbf{z})$ can be unbounded). We shall assume this for a while.

Taking $\sigma = \Phi_c[\rho]$, from (97) we have

$$\underline{p}(\mathbf{z}) = \frac{1}{c^{2s}} \text{Tr} \rho D(c^{-1}\mathbf{z}) \rho_0 D^*(c^{-1}\mathbf{z}) = \frac{1}{c^{2s}} p_\rho(c^{-1}\mathbf{z}).$$

while

$$\bar{p}(\mathbf{z}) = \text{Tr} \rho_{\mathbf{z}} \Phi_c[\rho] = \int_{\mathbf{w}} \underline{p}(\mathbf{w}) \text{Tr} \rho_{\mathbf{z}} \rho_{\mathbf{w}} \frac{d^{2s}\mathbf{w}}{\pi^s}. \quad (100)$$

We use the well-known formula, see e.g. [44], [33],

$$\text{Tr} \rho_{\mathbf{z}} \rho_{\mathbf{w}} = \exp[-|\mathbf{z} - \mathbf{w}|^2].$$

By introducing the probability density of a normal distribution

$$q_c(\mathbf{z}) = \frac{c^{2s}}{\pi^s} \exp(-c^2|\mathbf{z}|^2)$$

tending to δ -function when $c \rightarrow \infty$ and substituting this into (100), we have

$$\begin{aligned} \bar{p}(\mathbf{z}) &= \int d^{2s}\mathbf{w} \underline{p}(\mathbf{w}) q_1(\mathbf{z} - \mathbf{w}) \\ &= \int d^{2s}\mathbf{w}' p_\rho(\mathbf{w}') q_1(\mathbf{z} - c\mathbf{w}') \\ &= \frac{1}{c^{2s}} p_\rho * q_c(c^{-1}\mathbf{z}). \end{aligned} \quad (101)$$

With the change of the integration variable $c^{-1}\mathbf{z} \rightarrow \mathbf{z}$, the inequalities (99) become

$$\int_{\mathbf{z}} f(c^{-2s} p_\rho(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s} \leq c^{-2s} \text{Tr} f(\Phi_c[\rho]) \leq \int_{\mathbb{C}^s} f(c^{-2s} p_\rho * q_c(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s},$$

Substituting $\rho = \rho_{\mathbf{w}}$, we have

$$\int_{\mathbf{z}} f(c^{-2s} p_{\rho_{\mathbf{w}}}(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s} \leq c^{-2s} \text{Tr} f(\Phi_c[\rho_{\mathbf{w}}]) \leq \int_{\mathbf{z}} f(c^{-2s} p_{\rho_{\mathbf{w}}} * q_c(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s}.$$

Combining the last two displayed formulas with (98) we obtain

$$\begin{aligned} & \int_{\mathbf{z}} g(p_{\rho}(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s} - \int_{\mathbb{C}^s} g(p_{\rho_{\mathbf{w}}}(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s} \\ & \geq \int_{\mathbf{z}} g(p_{\rho}(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s} - \int_{\mathbf{z}} g(p_{\rho} * q_c(\mathbf{z})) \frac{d^{2s}\mathbf{z}}{\pi^s}, \end{aligned} \quad (102)$$

where we denoted $g(x) = f(c^{-2s}x)$, which is again a concave function. Moreover, arbitrary concave polygonal function g on $[0, 1]$, satisfying $g(0) = 0$, can be obtained in this way by defining

$$f(x) = \begin{cases} g(c^{2s}x), & x \in [0, c^{-2s}] \\ g(1) + g'(1)(x - c^{-2s}), & x \in [c^{-2s}, \infty) \end{cases},$$

hence (102) holds for any such function. Then the right hand side of the inequality (102) tends to zero as $c \rightarrow \infty$. Indeed, for polygonal function $|g(x) - g(y)| \leq \varkappa |x - y|$, and the asserted convergence follows from the convergence $p_{\rho} * q_c \rightarrow p_{\rho}$ in L_1 : if $p(\mathbf{z})$ is a bounded continuous probability density, then

$$\lim_{c \rightarrow \infty} \int_{\mathbf{z}} |p * q_c(\mathbf{z}) - p(\mathbf{z})| d^{2s}\mathbf{z} = 0.$$

Thus we obtain (96) for the concave polygonal functions f . But for arbitrary continuous concave f on $[0, 1]$ there is a monotonously nondecreasing sequence of concave polygonal functions f_n converging to f . Applying Beppo-Levy's theorem, we obtain the statement. ■

4 Appendix

Consider a gauge-covariant channel Φ such that the matrices $\mathbf{K}^*\mathbf{K}$ and $\boldsymbol{\mu}$ commute (in particular, this condition is satisfied by extreme amplifiers and attenuators). These channels are diagonalizable in the following sense. We have

$$\mathbf{K} = \mathbf{V}_A \mathbf{K}_d \mathbf{V}_B, \quad \boldsymbol{\mu} = \mathbf{V}_B^* \boldsymbol{\mu}_d \mathbf{V}_B,$$

where $\mathbf{V}_A, \mathbf{V}_B$ are unitaries and $\mathbf{K}_d, \boldsymbol{\mu}_d$ are diagonal (rectangular) matrices with nonnegative values on the diagonal. Then $\mathbf{K}^* \mathbf{K} = \mathbf{V}_B^* \mathbf{K}_d^2 \mathbf{V}_B$, and

$$\Phi[\rho] = U_B \Phi_d [U_A \rho U_A^*] U_B^*, \quad (103)$$

where U_A, U_B are canonical unitary (“metaplectic” [2]) transformations acting on $\mathcal{H}_A, \mathcal{H}_B$ such that

$$U_B^* D_B(\mathbf{z}) U_B = D_B(\mathbf{V}_B \mathbf{z}), \quad U_A^* D_A(\mathbf{z}) U_A = D_A(\mathbf{V}_A \mathbf{z}),$$

To describe the action of “diagonal” channel Φ_d in more detail, we have to consider separately the cases $s_A = s_B$, $s_A \leq s_B$ and $s_A > s_B$.

In the case $s_A = s_B$ we have

$$\mathbf{K}_d = \text{diag} [k_j]_{j=1, \dots, s_B}; \quad \boldsymbol{\mu}_d = \text{diag} [\mu_j]_{j=1, \dots, s_B}.$$

Then $\Phi_d = \otimes_{j=1}^{s_B} \Phi_j$, where, in self-explanatory notations,

$$\Phi_j^* [D_j(z_j)] = D_j(k_j z_j) \exp(-\mu_j |z_j|^2). \quad (104)$$

In the case $s_A < s_B$

$$\mathbf{K}_d = \begin{bmatrix} \text{diag} [k_j]_{j=1, \dots, s_A} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ denotes block of zeroes of the size $(s_B - s_A) \times s_A$. Then

$$\Phi_d[\rho] = \otimes_{j=1}^{s_A} \Phi_j[\rho] \otimes \rho_0^{[s_A+1, \dots, s_B]},$$

where for $j = 1, \dots, s_A$ the one-mode channels Φ_j are given by (104), and $\rho_0^{[s_A+1, \dots, s_B]}$ is the vacuum state of the modes $s_A + 1, \dots, s_B$.

In the case $s_A > s_B$

$$\mathbf{K}_d = \begin{bmatrix} \text{diag} [k_j]_{j=1, \dots, s_B} & \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ denotes block of zeroes of the size $s_B \times (s_A - s_B)$, and

$$\Phi_d[\rho] = \left(\otimes_{j=1}^{s_A} \Phi_j \right) [\text{Tr}_{s_B+1, \dots, s_A} \rho],$$

where $\text{Tr}_{s_B+1, \dots, s_A}$ denotes partial trace over the last $s_A - s_B$ modes of the operator ρ .

There is a similar reduction to the diagonal form for gauge-contravariant channels.

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